AMHERST COLLEGE

Department of Mathematics and Computer Science

COMPREHENSIVE EXAMINATION

Mathematics 11, 12, 13, 25

2:00 p.m. Friday, January 29, 1999

Seeley Mudd 205

Comprehensive Examination [Mathematics 11, 12, 13, 25] January 29, 1999

1. Evaluate each limit or determine that it doesn't exist.

   (a) $\lim_{x \to 1} \frac{2^x - 2}{\ln x}$

   (b) $\lim_{b \to +\infty} \int_1^b x e^{-2x}\, dx$

   (c) $\lim_{x \to 0} (\sin x)^x$

2. Evaluate each integral.

   (a) $\int_1^2 \frac{1}{x^2 + 2x}\, dx$

   (b) $\int_0^{\pi/2} \cos^3 x\, dx$

   (c) $\int_0^1 \sqrt{1 - x^2}\, dx$

3. (a) Let n be a positive integer. Derive a formula for $\int (\ln x)^n\, dx$ in terms of $\int (\ln x)^{n-1}\, dx$.

   (b) Use part (a) to compute $\int (\ln x)^4\, dx$.

4. (a) State the ϵ-δ definition of $\lim_{x \to a} F(x) = L$.

   (b) Give an ϵ-δ proof that $\lim_{x \to 2} 3x - 7 = -1$

5. In each case determine whether the given series converges absolutely, converges conditionally, or diverges.

   (a) $\sum_{n=1}^{\infty} \frac{n^2}{n^3 + \sin(n)}$

   (b) $\sum_{n=1}^{\infty} (-1)^n n e^{-n^2}$

   (c) $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n! + 1}$

6. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{(3n)!}{(n!)^3} x^n$

7. Consider the double integral $\int_0^2 \int_0^x x \, dy \, dx$

   (a) This integral represents the volume under some surface over some region in the x-y plane. What is the surface? What is the region?

   (b) Express the integral in polar coordinates.

8. Find a function f such that $\nabla f = (2xy - \cos x) \ \vec{i} \ + (x^2 + 2y \sin(y^2)) \ \vec{j}$.

9. Find the critical points of $x^2y - x^2 - y^2$ and classify them as to local maximum, local minimum, or saddle point.

10. (a) Define what it means for a function $f(x, y)$ to be differentiable at a point (x_0, y_0).

    (b) State a theorem whose conclusion is that a function is differentiable.

    (c) Give an ϵ-δ proof that $f(x, y) = x^2 + y^2$ is differentiable at $(0, 0)$.

11. Suppose that $T : V \to W$ is a linear map between vector spaces. If $v_1, \ldots, v_n$ are vectors in V such that $T(v_1), \ldots, T(v_n)$ are distinct and linearly independent in W, then prove that $v_1, \ldots, v_n$ are linearly independent in V.

12. Suppose that $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear map which satisfies $T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

    (a) Find a 2×2 matrix A such that $T\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$ for all vectors $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2$.

    (b) Is T an isomorphism? Justify your answer.

13. Let

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 2 & -1 & \lambda \\ 3 & 0 & 2 \end{pmatrix}$$

    (a) Find all eigenvalues of A.

    (b) For which value of λ is A diagonalizable? Justify your answer.

AMHERST COLLEGE

Department of Mathematics and Computer Science

COMPREHENSIVE EXAMINATION: MATHEMATICS 26

January 29, 1999

Work the following four problems. Record your answers in the blue book provided. PLEASE SHOW ALL YOUR WORK.

1. Suppose $\phi : G \to G'$ is a homomorphism of groups and N' is a normal subgroup of G'. Let $N = \{a \in G : \phi(a) \in N'\}$. Show that N is a normal subgroup of G.

2. Let G be a finite group. Suppose x and y are distinct elements of order two in G such that $xy = yx$. Show that the order of G is divisible by 4.

3. Let R be a commutative ring with a multiplicative identity, and let I be an ideal of R. Show that R/I is a field if and only if I is a maximal ideal of R.

4. Let k be a field. Show that a cubic polynomial $f(x) \in k[x]$ is irreducible in $k[x]$ if and only if $f(x)$ has no roots in k.

AMHERST COLLEGE

Department of Mathematics and Computer Science

COMPREHENSIVE EXAMINATION: MATHEMATICS 28

March 26, 1999

Do the following three problems. Record your answers in the blue book provided. PLEASE SHOW ALL YOUR WORK.

1. (a) Complete the following definition: Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of real valued functions defined on a set A of real numbers. The infinite series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on A to the function $g(x)$ if...

   (b) State the Weierstrass M-test.

   (c) Prove the Weierstrass M-test.

2. Let the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ of real numbers converge to real numbers A and B respectively. Using the definition of convergence of a sequence, give a rigorous proof that the sequence $\{a_n + b_n\}_{n=1}^{\infty}$ converges to $A + B$.

3. (a) State the Completeness Axiom for the real numbers.

   (b) Let a and b be positive real numbers. Prove that there exists an integer n such that $na > b$.

AMHERST COLLEGE

Department of Mathematics and Computer Science

COMPREHENSIVE EXAMINATION

Mathematics 11, 12, 13, 25

2:00 p.m. Friday, January 30, 1998

Seeley Mudd 205

Comprehensive Examination [Mathematics 11, 12, 13, 25] January 30, 1998

1. Evaluate each integral:

   (a) $\int \ln(1/x)\,dx$

   (b) $\int_0^2 \sqrt{4-x^2}\,dx$

   (c) $\int_4^{+\infty} \frac{1}{x^2-5x+6}\,dx$

2. Evaluate each limit or determine that it does not exist:

   (a) $\lim_{x\to+\infty} (1-2/x)^{3x}$

   (b) $\lim_{x\to 0} \frac{x-\sin^{-1} x}{x^3}$

   (c) $\lim_{x\to-\infty} \frac{\sqrt{4x^2+1}}{3x+1}$

3. Let f be a real-valued function defined on an open interval containing the point x_0.

   (a) Define what it means for f to be **differentiable** at x_0.

   (b) Prove that if f is differentiable at x_0, then f is continuous at x_0.

   (c) Is the converse of (b) true? Give a proof or counterexample.

4. Let C be the curve given by parametric equation $x = \cos^3 t$; $y = \sin^3 t$ for $0 \le t \le \pi/2$.

   (a) Find dy/dx at the point where $t = \pi/6$.

   (b) Find the length of C.

5. (a) In each case determine whether the given series converges or diverges. Give reasons.

i) $\sum_{n=1}^{\infty} n\sin(1/n)$

ii) $\sum_{n=1}^{\infty} \frac{\cos n}{n^2+1}$

iii) $\sum_{n=1}^{\infty} \frac{(-3)^n (n!)^2}{(2n)!}$

(b) Find **all** values of x for which the series $\sum_{n=1}^{\infty} \frac{2^n(2x-3)^n}{\sqrt{2n+1}}$ converges. Give reasons.

6. A rectangular box with edges parallel to the coordinate axes is inscribed in the ellipsoid $9x^2 + 3y^2 + z^2 = 9$. What is the greatest possible volume of such a box?

7. Let $F(x, y, z) = z - xy - x$.

(a) Find the directional derivative of F at $(1,2,3)$ in the direction from $(1,2,3)$ to $(2,4,1)$.

(b) What is the least possible directional derivative of F at $(1,2,3)$?

(c) Let S be the surface $F(x,y,z) = 1$. Is the vector $\vec{v} = (1,2,3)$ perpendicular to the tangent plane to S at $(\frac{1}{2}, 2, \frac{5}{2})$? Explain your reasoning.

8. Let R be the solid bounded above by the sphere $x^2 + y^2 + z^2 = 12$ and below by the cone $\sqrt{3}\,z = \sqrt{x^2+y^2}$. Suppose that R has density $d(x,y,z) = z$. Set up three triple integrals giving the mass of R, one in rectangular, one in cylindrical, and one in spherical coordinates. Then evaluate one of your three integrals.

9. (a) Sketch a graph of the polar-coordinate curve $r = 1 + \cos\theta$ for $0 \leq \theta \leq 2\pi$.

(b) Evaluate the line integral $\int_C (x^3 + y^3 - y)\,dx + (3xy^2 + y^3)\,dy$, where C is the curve of (a) oriented counter-clockwise.

10. Let V and W be vector spaces over a field F and let T be a linear transformation from V to W.

(a) Prove that the nullspace $N(T)$ of T (also called the kernel of T) is a subspace of V.

(b) Prove that T is one-one if and only if $N(T) = \{\vec{0}\}$.

(c) Show that if T is one-one and $\{\vec{x}_1, \ldots, \vec{x}_n\}$ is a set of n linearly independent vectors in V, then $\{T(\vec{x}_1), \ldots, T(\vec{x}_n)\}$ is linearly independent in W.

11. The set of solutions of the homogeneous system of equations

$$\begin{cases} x + 2y \phantom{{}+z} + 3w &= 0 \\ x + 2y + z + 2w &= 0 \\ x + 2y + 3z &= 0 \end{cases}$$

forms a subspace of $\mathbf{R}^4$. Find a basis for this subspace.

12. Define the linear transformation T from $\mathbf{R}^2$ to $\mathbf{R}^2$ by $T(x, y) = (y, -4x + 4y)$.

(a) Is T invertible? If so, find the formula for $T^{-1}(x, y)$.

(b) Find the eigenvalues of T.

(c) Prove or disprove: T is diagonalizable.

AMHERST COLLEGE

Department of Mathematics and Computer Science

COMPREHENSIVE EXAMINATION: MATHEMATICS 26

January 30, 1998

Work the following four problems. Record your answers in the blue book provided. PLEASE SHOW ALL YOUR WORK.

1. Let S_3 denote the symmetric group on $\{1, 2, 3\}$ and let $\sigma, \tau \in S_3$ denote the permutations

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

   Consider the subgroups $H = < \sigma >$ and $K = < \tau >$ of S_3 generated by σ and τ respectively. Are H and K normal subgroups of S_3? Prove your answer for each.

2. Let G be a group and let $I(G) = \{x \in G \mid x = x^{-1}\}$.

   (a) Show that if G is abelian, then $I(G)$ is a subgroup of G.

   (b) Give an example of a group G for which $I(G)$ is not a subgroup of G.

   (c) Show that if G is finite and $I(G) \neq \{e\}$, then G must have even order.

3. Recall that if R is a commutative ring and $a \in R$, then $(a) = \{ra \mid r \in R\}$ denotes the principal ideal of R generated by a.

   (a) Show that (5) is a maximal ideal of the integers $\mathbf{Z}$.

   (b) Show that (5) is not a maximal ideal of the Gaussian integers $\mathbf{Z}[i] = \{a + bi \mid a, b \in \mathbf{Z}\}$.

4. Let R be a commutative ring with identity 1, and let I and J be ideals of R. Suppose there are elements $x \in I$ and $y \in J$ such that $x + y = 1$.

   (a) Show that $I + J = R$.

   (b) Define $\phi : I \to R/J$ by $\phi(a) = a + J$ for all $a \in I$. Show that if $I \cap J = (0)$, then ϕ is an isomorphism of I onto R/J.

AMHERST COLLEGE

Department of Mathematics and Computer Science

COMPREHENSIVE EXAMINATION: MATHEMATICS 28

January 30, 1998

Work the following three problems. Record your answers in the blue book provided. PLEASE SHOW ALL YOUR WORK.

1. (a) State the Completeness Axiom for the real numbers.

   (b) Prove that the square root of 2 is not a rational number.

   (c) Let x be a real number. Give an example of a sequence of irrational numbers which converges to x. (You may use the result of part (b) even if you did not do that part.)

2. State and prove a theorem having the following hypothesis: Let $y = f(x)$ define a continuous function for $a \leq x \leq b$.

3. Consider the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

   (a) Prove that this series converges uniformly on $(-77, 66]$.

   (b) Using the definition of uniform convergence, explain as best you can why this series fails to converge uniformly on $(-\infty, 0]$. (Hint: You may wish to identify the function represented by this series.)

AMHERST COLLEGE

Department of Mathematics and Computer Science

COMPREHENSIVE EXAMINATION

Mathematics 11, 12, 13, 25

2:00 p.m. Friday, January 31, 1997

Seeley Mudd 205

Comprehensive Examination [Mathematics 11, 12, 13, 25] January 31, 1997

1. Evaluate the following limits.

   (a) $\displaystyle\lim_{x\to 0} \frac{xe^x - \ln(1+x)}{x^2}$

   (b) $\displaystyle\lim_{x\to +\infty} (1 - e^{-x})^{e^x}$

2. Evaluate the following derivatives.

   (a) $\displaystyle\frac{d}{dx}\int_0^x \sec^3\theta\, d\theta$

   (b) $\displaystyle\frac{d}{dt}F(f(t), g(t))$, where $F(x,y)$ is a differentiable function of x, y and $f(t)$, $g(t)$ are differentiable functions of t

3. Evaluate the following integrals.

   (a) $\displaystyle\int_2^\infty \frac{dx}{x^2-1}$

   (b) $\displaystyle\int_0^\pi \int_x^\pi \frac{\sin y}{y}\, dy\, dx$

   (c) $\displaystyle\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} x^2\, dy\, dx$

4. (a) State the Mean Value Theorem.

   (b) Let f be a differentiable function on the interval (a,b) with the property that $f'(c) > 0$ for all c in (a,b). Use the Mean Value Theorem to prove rigorously that f is increasing on (a,b).

5. For each of the following series, determine if it converges or diverges. Give reasons for your answers.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^2+1}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{2^n n!}$

6. Find all values of x for which the series $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n}}{2^n\sqrt{n}}$ converges.

7. Let $f(x,y) = \begin{cases} \frac{x^2y^2\cos x}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

(a) Compute $f_x(0,0)$ and $f_y(0,0)$.

(b) Prove that f is differentiable at $(0,0)$.

(c) Is f continuous at $(0,0)$? Explain your reasoning.

8. Given point $P = (1,2)$ and $Q = (2,1)$, let γ be a path in the plane not going through $(0,0)$ which connects P to Q.

(a) Explain why the line integral

$$\int_\gamma \frac{10x}{(x^2+y^2)^2}dx + \frac{10y}{(x^2+y^2)^2}dy$$

gives the same answer for all possible γ.

(b) Find the value of the line integral in part (a).

9. Consider the region in 3-dimensional space bounded above by the hemisphere $z = \sqrt{8 - x^2 - y^2}$ and bounded below by $z = \sqrt{x^2+y^2}$.

(a) Express the volume of this region using cartesian coordinates, cylindrical coordinates and spherical coordinates.

(b) Evaluate one of the integrals found in part (a).

10. (a) Define what it means for a real number $\lambda \in \mathbf{R}$ to be an eigenvalue of an $n \times n$ matrix $A \in M_{n\times n}(\mathbf{R})$.

(b) Find all eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

11. Let $\vec{v}_1, \ldots, \vec{v}_k$ be linearly independent vectors in a vector space V, and let $\vec{v} \in V$. Prove that $\vec{v}, \vec{v}_1, \ldots, \vec{v}_k$ are linearly independent if and only if $\vec{v} \notin \text{Span}(\vec{v}_1, \ldots, \vec{v}_k)$.

12. Let $L : \mathbf{R}^3 \to \mathbf{R}^2$ be a linear map.

(a) Can L be one-to-one? Explain your reasoning.

(b) Describe how you would construct a 2×3 matrix A with the property that $L(\vec{v}) = A\vec{v}$ for all $\vec{v} \in \mathbf{R}^3$.

AMHERST COLLEGE

Department of Mathematics and Computer Science

COMPREHENSIVE EXAMINATION: MATHEMATICS 26

January 31, 1997

Work the following four problems. Record your answers in the blue book provided. PLEASE SHOW ALL YOUR WORK.

1. Suppose G is a nontrivial group (i.e., $G \neq \{e\}$) whose only subgroups are the trivial group $\{e\}$ and itself. Show that G is a cyclic group of prime order.

2. Suppose that σ is a permutation in the alternating group A_{10} given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 7 & 2 & 6 & 10 & 1 & 5 & & & 3 \end{pmatrix}$$

where the images of 8 and 9 have been lost. Determine the images of 8 and 9 under σ. What is the order of σ?

3. Let R be a commutative ring and let I be a proper ideal of R. I is said to be a **prime ideal** of R if, for all a, $b \in R$, $ab \in I$ implies $a \in I$ or $b \in I$. Prove that R/I is an integral domain if and only if I is a prime ideal. (You may assume that R/I is a commutative ring in your proof.)

4. Let $\mathbf{R}[x]$ denote the ring of polynomials in x with real coefficients, and let $\mathbf{C}$ denote the field of complex numbers. Define a map $\phi : \mathbf{R}[x] \longrightarrow \mathbf{C}$ by $\phi(p(x)) = p(i)$, where i is the usual complex number satisfying $i^2 = -1$. You may assume that ϕ is a ring homomorphism.

   (a) Show that the kernel of ϕ is (x^2+1), the principal ideal of $\mathbf{R}[x]$ generated by the polynomial $x^2 + 1$.

   (b) Show that $\mathbf{R}[x]/(x^2 + 1)$ is isomorphic to $\mathbf{C}$.

AMHERST COLLEGE

Department of Mathematics and Computer Science

COMPREHENSIVE EXAMINATION: MATHEMATICS 28

January 31, 1997

Work the following three problems. Record your answers in the blue book provided. PLEASE SHOW ALL YOUR WORK.

1. (a) Complete the following definition: The real number P is an **accumulation point** (also known as a **cluster point**) of the set A of real numbers if...

   (b) State the Bolzano-Weierstrass Theorem.

   (c) Complete the following definition: Let f be a real valued function defined on the set A of real numbers. Then f is **uniformly continuous** on A if...

2. (a) Prove that the *sequence* $\{e^{-nx}\}_{n=0}^{\infty}$ converges **uniformly** on $(1, \infty)$.

   (b) Prove that *series* $\sum_{n=0}^{\infty} x^n$ does **not** converge uniformly on $(0, 1)$.

3. The goal of this problem is to prove that if $f : [a, b] \to R$ is continuous and $f(a) > 0$, $f(b) < 0$, then $f(c) = 0$ for some c in (a, b).

   (a) Explain how this result easily follows from the Intermediate Value Theorem.

   (b) Give a direct proof of the result which uses **only** the definition of continuity and the properties of the real numbers. Hint: Prove carefully that the least upper bound of the set $\{x \in [a, b] : f(x) > 0\}$ exists and is in (a, b). Let c denote this least upper bound. Then prove carefully that $f(c) = 0$.

AMHERST COLLEGE

Department of Mathematics and Computer Science

COMPREHENSIVE EXAMINATION

Mathematics 11, 12, 13, 25

2:00 p.m. Friday, February 2, 1996

Seeley Mudd 205

Comprehensive Examination [Mathematics 11, 12, 13, 25] February 2, 1996

1. (a) Use L'Hôpitals rule to evaluate $\lim_{x \to 0} \frac{x \cos x - \sin x}{x^3}$.

   (b) Use the power series expansions of $\sin x$ and $\cos x$ to evaluate $\lim_{x \to 0} \frac{x \sin x - \cos x}{x^3}$.

   (c) Evaluate $\lim_{x \to -\infty} \frac{\sqrt{x^2 - 1}}{2x + 1}$.

   (d) Evaluate $\lim_{x \to +\infty} (x^a + 1)^{1/\ln x}$, where $a > 0$.

2. Evaluate the following integrals.

   (a) $\int x \tan^{-1} x \, dx$

   (b) $\int (F(x))^2 (\ln x)^2 \, dx$, where $F(x) = \int_1^x (\ln t)^2 \, dt$

   (c) $\int_C (1 - xy) \, dx + (x + y^2) \, dy$, where C is the boundary (oriented counterclockwise) of the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$.

3. (a) State the Mean Value Theorem.

   (b) Use the Mean Value Theorem on the interval $[1, x]$ to show that $e^x > ex$ when $x > 1$.

4. For each of the following infinite series, determine if it converges absolutely, converges conditionally, or diverges. Give reasons for your answers.

   (a) $\sum_{n=0}^{\infty} \frac{(2n)!}{3^n (n!)^2}$

   (b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

   (c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{n^2 + 1}$

5. For what values of x does the following series converge? Justify your answer.

$$\sum_{n=0}^{\infty} \frac{\cos^n x}{n + 1}$$

6. Let $f(x,y)$ be defined by

$$f(x,y) = \begin{cases} \frac{2xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(a) Show that $f_x(0,0)$ and $f_y(0,0)$ exist.

(b) Is $f(x,y)$ differentiable at $(0,0)$? Give full reasons for your answer.

7. Let $F(x,y) = x^2 + y^2 - 2x - 2y$.

(a) Find the critical points of $F(x,y)$ and classify them as to local maximum, local minimum or saddle point.

(b) Find the absolute minimum and maximum values of $F(x,y)$ subject to the constraint $x^2 + y^2 = 8$.

(c) By combining parts a and b, determine the absolute minimum and maximum values of $F(x,y)$ in the region $x^2 + y^2 \leq 8$. Explain your reasoning.

8. Find the volume of the region inside the cylinder $x^2 + y^2 = a^2$ which lies between the planes $z = 0$ and $z = x + a$. (Here, $a > 0$ is a constant.)

9. Let V and V' be vector spaces over a field F and let $T : V \to V'$ be a linear transformation. If $W' \subseteq V'$, let $W = \{\vec{v} \in V : T(\vec{v}) \in W'\}$. Show that if W' is a subspace of V', then W is a subspace of V.

10. Let $V = M_{2\times 2}(\mathbf{R})$ and consider the subspaces $W_1 = \{A \in V : A \text{ is symmetric}\}$, $W_2 = \{A \in V : tr(A) = 0\}$ and $W_3 = \{A \in V : A \text{ is a diagonal matrix}\}$.

(a) Find bases for W_1, W_2 and W_3. You do not need to prove that you have found a basis.

(b) Carefully compute $\dim(W_1 + W_3)$ and $\dim(W_2 + W_3)$.

11. Suppose the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ satisfies $T(1,1) = (3,2)$ and $T(2,1) = (5,4)$.

(a) Find a formula for $T(x,y)$.

(b) Is T invertible? If so, find $T^{-1}(x,y)$.

12. Define $T : \mathbf{R}^3 \to \mathbf{R}^3$ by $T(x,y,z) = (3x + 3y, 3x + 3y, -3x + y + 4z)$.

(a) Compute the rank $r(T)$ and the nullity $n(T)$.

(b) Prove or disprove: T is diagonalizable.

AMHERST COLLEGE

Department of Mathematics and Computer Science

COMPREHENSIVE EXAMINATION: MATHEMATICS 28

February 2, 1996

Work the following four problems. Record your answers in the blue book provided. PLEASE SHOW ALL YOUR WORK.

1. (a) State the Completeness Axiom for the real numbers.

   (b) State the Bolzano-Weierstrass Theorem.

   (c) State the Intermediate Value Theorem for Continuous Functions.

   (d) Complete the following definition: Let f be a real-valued function defined on the set A of real numbers. If $a \in A$, then f is **continuous** at a if and only if ....

2. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be convergent sequences of real numbers, with respective limits A and B. Using the definition of convergence of a sequence, prove that the sequence $\{a_n + b_n\}_{n=1}^{\infty}$ converges to $A + B$.

3. Let f be a real-valued, continuous function on the closed, bounded interval $[a, b]$. Assuming that f is bounded on $[a, b]$, prove that f takes on a maximum on $[a, b]$, i.e., that there must exist $c \in [a, b]$ such that $f(c) \geq f(x)$ for all $x \in [a, b]$.

4. Consider the sequence of functions $\{x^n\}_{n=1}^{\infty}$.

   (a) Prove that this sequence converges **uniformly** on $[0, \frac{1}{2}]$.

   (b) Prove that this sequence does **not** converge uniformly on $[0, 1]$.

AMHERST COLLEGE

Department of Mathematics and Computer Science

COMPREHENSIVE EXAMINATION

Mathematics 11, 12, 13, 25

2:00 p.m. Friday, February 3, 1995

Seeley Mudd 205

Instructions: Work all the problems in this section. Record your solutions in the blue book(s) provided. SHOW ALL WORK.

Comprehensive Examination [Mathematics 11, 12, 13, 25] February 3, 1995

1. (a) Find a positive rational number and a positive irrational number both smaller than 0.00001.

   (b) Find the solution set for $\frac{x+5}{2x-1} \leq 0$

2. Find the following limits or show that no limit exists.

   (a) $\lim_{x\to 2} \frac{1-\frac{2}{x}}{x^2-4}$

   (b) $\lim_{x\to 0} \frac{\tan x}{\sin 2x}$

   (c) $\lim_{t\to 2^-} ([\![t]\!] - t)$, where $[\![t]\!]$ is the greatest integer less than or equal to t.

   (d) $\lim_{(x,y)\to(0,0)} \frac{xy+y^3}{x^2+y^2}$

3. (a) Find the equation of the tangent line to $x^2y^2 + 3xy = 10y$ at $(2,1)$.

   (b) Find the local extreme values of $f(x) = (\sin x)^{2/3}$ on $\left[-\frac{\pi}{6}, \frac{2\pi}{3}\right]$. For which values of x is f increasing, decreasing, concave up, concave down?

4. Evaluate:

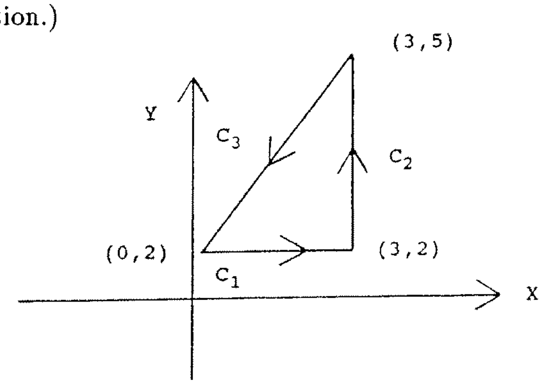
   (a) $\int_0^{\frac{\pi}{4}} \tan^5 x\, dx$

   (b) $\int_1^2 \ln x\, dx$

   (c) $\int \frac{1}{a^2-x^2}\, dx,\ a>0$

   (d) $\int_0^4 \int_{\frac{x}{2}}^2 e^{y^2}\, dy\, dx$ (Hint: Change the order of integration.)

   (e) $\int_C xy^2\, dx + xy^2\, dy$ along $C = C_1 \cup C_2 \cup C_3$, where



5. (a) Do the following series converge absolutely, converge conditionally, or diverge? Give reasons.

   (i) $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \dots$

   (ii) $\sum_{k=1}^{\infty} \ln \frac{k}{k+1}$

   (iii) $\sum_{k=1}^{\infty} kr^k, \; |r| < 1.$

   (iv) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{e^n}$

   (b) Find the convergence set for the series $1 + (x+2) + \frac{(x+2)^2}{2!} + \frac{(x+2)^3}{3!} + \dots$

   (c) Find the Taylor series in $(x-a)$ through the term $(x-a)^3$ for $\cos x$, where $a = \frac{\pi}{3}$.

6. (a) Find the area enclosed by the graph of the polar equation $r = 4\sin 3\theta$.

   (b) Find the volume of the tetrahedron bounded by the coordinate planes and the plane $3x + 6y + 4z - 12 = 0$.

7. The temperature of a ball centered at the origin is given by $T(x,y,z) = \frac{200}{5+x^2+y^2+z^2}$.

   (a) By inspection decide where the ball is hottest.

   (b) Find a vector pointing in the direction of greatest increase in temperature at $(1,-1,1)$.

   (c) Does the vector in part (b) point toward the point where the ball is hottest?

8. Determine if $\overrightarrow{F} = (4x^3 + 9x^2y^2)\,\vec{i} + (6x^3y + 6y^5)\,\vec{j}$ is conservative, and if so find a function f of which it is the gradient.

9. Let V be a vector space over a field F and let W_1 and W_2 be subspaces of V.

   (a) Show that $W_1 + W_2$ is a subspace of V.

   (b) Give an example to show that $W_1 \cup W_2$ need not be a subspace of V.

10. Is the real matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$ invertible? If so, find its inverse.

11. Define $T : \mathbf{R}^3 \to \mathbf{R}^3$ by $T(x, y, z) = (3x - 2y,\ -2x + 3y,\ 5z)$. Find all the eigenvalues of T and determine whether or not T is diagonalizable.

12. Let T be a linear transformation on a finite-dimensional vector space V over $\mathbf{R}$ and suppose that $T^2 = T$.

(a) Show that $V = N(T) \oplus R(T)$.

(b) Show that $T + I$ is invertible, where I is the identity transformation on V.

AMHERST COLLEGE

Department of Mathematics and Computer Science

COMPREHENSIVE EXAMINATION: MATHEMATICS 26

February 3, 1995

Work the following four problems. Record your answers in the blue book provided.

PLEASE SHOW ALL YOUR WORK.

1. Let $\phi: G \longrightarrow G'$ be a homomorphism of groups and suppose $x \in G$ has order $n \geq 1$.
   (a) Show that the order of $\phi(x)$ divides n.
   (b) Prove that if the order of G' is relatively prime to n, then x is in the kernel of ϕ.

2. Let G be a group and define $Z(G) = \{g \in G \mid ga = ag \text{ for all } a \in G\}$.
   (a) Show that $Z(G)$ is a subgroup of G.
   (b) Show that the subgroup $Z(G)$ is normal in G.
   (c) Prove that if the quotient group $G/Z(G)$ is cyclic, then G is abelian.

3. Suppose R and R' are rings and $\psi: R \longrightarrow R'$ is a ring homomorphism with kernel K. Suppose A' is a subring of R' and let $A = \{a \in R \mid \psi(a) \in A'\}$.
   (a) Show that A is a subring of R and that A contains K.
   (b) Prove that if A' is an ideal of R', then A is an ideal of R.

4. Let F be a field and let $p(x)$ be a polynomial in $F[x]$ of degree 3. Write $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ with $a_0, a_1, a_2, a_3 \in F$. Prove that if there is no element $r \in F$ such that $p(r) = a_3r^3 + a_2r^2 + a_1r + a_0 = 0$, then $p(x)$ is irreducible in $F[x]$.

AMHERST COLLEGE

Department of Mathematics and Computer Science

COMPREHENSIVE EXAMINATION: MATHEMATICS 28

February 3, 1995

Work the following four problems. Record your answers in the blue book provided. PLEASE SHOW ALL YOUR WORK.

1. Let f and g be functions, $f, g : \mathbf{R} \to \mathbf{R}$, and let $a \in \mathbf{R}$. Suppose f is continuous at a and g is continuous at $f(a)$. Prove that $g \circ f(x) = g(f(x))$ is continuous at a.

2. Let

$$S_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}, \qquad n = 1, 2, 3, \cdots$$

   Prove that $\{S_n\}_{n=1}^{\infty}$ converges and $\lim_{n \to \infty} S_n \leq \frac{1}{2}$.

3. State and prove the Bolzano-Weierstrass Theorem.

4. Let

$$f(x) = \sum_{n=0}^{\infty} e^{-nx} x^n, \qquad (0 \leq x \leq 10)$$

   (a) Does this series converge uniformly on $[0, 10]$?

   (Hint: Find the maximum value of xe^{-x}.)

   (b) Find the sum of the series representing $f(x)$.