AMHERST COLLEGE

Department of Mathematics

COMPREHENSIVE EXAMINATION

Calculus and Linear Algebra

2:00pm Friday, January 29, 2010

Seeley Mudd 206

There are 13 problems (totaling 150 points) on this portion of the examination. Record your answers in the blue book provided. **Show all of your work.**

1. (15 points) Evaluate the following limits:

   (a) $\lim_{x \to 0} (x + e^x)^{1/x}$

   (b) $\lim_{x \to 3} \left[\frac{1}{x-3} - \frac{3}{x^2 - 3x} \right]$

2. (15 points) Evaluate the following integrals:

   (a) $\int x \arctan x \, dx$

   (b) $\int \frac{x^3}{\sqrt{x^2+1}} \, dx$

3. (15 points) Determine whether the following series converge or diverge. Justify your answers.

   (a) $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

   (b) $\sum_{n=1}^{\infty} \frac{n^n}{(n!)^2}$

   (c) $\sum_{n=1}^{\infty} \arctan n$

4. (10 points) Find all values of x for which the following series converges. Justify your answer.

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{n 2^n}$$

5. (10 points) Find the value of the following infinite series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n}}{(2n+1)!}$$

6. (15 points) Evaluate the following integrals:

(a) $\int_0^{\ln 10} \int_{e^x}^{10} \frac{1}{\ln y}\, dy\, dx.$

(b) $\int_C \cos(x^2)dx + (3xy^2 + x^3)dy$, where C is the circle $x^2 + y^2 = 4$, oriented counter-clockwise.

7. (10 points) Find the volume of the region that is inside the sphere $x^2 + y^2 + z^2 = 4$ and above the cone $z = \sqrt{3x^2 + 3y^2}$.

8. (10 points) Consider the function

$$f(x, y) = \begin{cases} \frac{x^3+4xy^2}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

(a) Find $f_x(0, 0)$ and $f_y(0, 0)$.

(b) Is f differentiable at (0,0)? Justify your answer.

9. (10 points) Find the point on the plane $2x - y + 2z = 16$ that is nearest the origin.

10. (10 points)

(a) State the Mean Value Theorem.

(b) Use the Mean Value Theorem to prove that if $f'(x) = 0$ at each $x \in [1, 3]$, then $f(x) = C$ for all $x \in [1, 3]$, where C is a constant.

11. (10 points) Let $\mathbf{A}$ be a square matrix and let α be a scalar that is NOT an eigenvalue of $\mathbf{A}$. Suppose that μ is an eigenvalue for the matrix $\mathbf{B} = (\mathbf{A} - \alpha\mathbf{I})^{-1}$ with corresponding eigenvector $\mathbf{v}$. Prove that $\mathbf{v}$ is also an eigenvector for $\mathbf{A}$ and find a formula for the corresponding eigenvalue of $\mathbf{A}$ in terms of μ and α.

12. (10 points) Let $T : V \to V$ be a linear transformation on a finite dimensional vector space V. Suppose T is one-to-one (injective). Prove that if $\{v_1, ..., v_n\}$ is a basis for V, then $\{T(v_1), ..., T(v_n)\}$ is also a basis for V.

13. (10 points) Let $T : P_2 \to \mathbb{R}^3$, where $P_2 = \{a + bt + ct^2 : a, b, c \in \mathbb{R}\}$, be defined by

$$T(p) = \begin{bmatrix} p(1) \\ p(1) \\ p'(1) \end{bmatrix}.$$

Here $p'(t)$ is the derivative of the polynomial $p(t)$. Determine the null space (kernel) and range of T.

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COMPREHENSIVE EXAMINATION: MATHEMATICS 26
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Do all four of the following problems. Write your solutions and all scratchwork in the blue book(s) provided. **Show all of your work, and justify your answers.**

1. (10 points) Let G and G' be groups, and let $\varphi : G \to G'$ and $\psi : G \to G'$ be homomorphisms. Define
$$H = \{g \in G \mid \varphi(g) = \psi(g)\}.$$
Prove that H is a subgroup of G.

2. (10 points) Let G be an abelian group. Let T be the set of elements of G that have finite order.
   (a) Show that T is a subgroup of G.
   (b) Show that in G/T, the only element of finite order is the identity.

3. (10 points) Let σ be the permutation $(4\ 2\ 1)(6\ 1\ 3\ 2)$ in S_6.
   (a) Write σ as a product of disjoint cycles in S_6.
   (b) What is the order of σ?
   (c) Is σ an even or an odd permutation?

4. (10 points) Let R be a commutative ring and $S \subseteq R$ a subset of R. Define the *annihilator* of S in R to be
$$\text{Ann}(S) = \{r \in R \mid rs = 0 \text{ for every } s \in S\}.$$
   (a) Show that $\text{Ann}(S)$ is an ideal of R.
   (b) If S and T are both subsets of R, show that
$$\text{Ann}(S) \cap \text{Ann}(T) = \text{Ann}(S \cup T).$$

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Do all four of the following problems. Write your solutions and all scratchwork in the blue book(s) provided. **Show all of your work, and justify your answers.**

1. (10 points)
   (a) State the Axiom of Completeness (also known as the Completeness Axiom or the Axiom of Continuity for the Real Numbers or Axiom C).
   (b) State the Heine-Borel Theorem.

2. (10 points) Use induction to prove that

$$\sum_{k=1}^{n} kx^{k-1} = \frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}$$

for all positive integers n.

3. (10 points)
   (a) State the definition of Cauchy sequence in $\mathbb{R}$.
   (b) Use the definition of Cauchy sequence to prove that every Cauchy sequence in $\mathbb{R}$ is bounded.

4. (10 points) Let $f_n(x) = \frac{nx}{1+nx}$ for $x \geq 0$.
   (a) State the function f to which the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise.
   (b) Prove that $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $[1, \infty)$.
   (c) Explain why $\{f_n\}_{n=1}^{\infty}$ does NOT converge uniformly on $[0, \infty)$.