Math 111, Introduction to the Calculus, Fall 2011
Midterm I Practice Exam 1 Solutions

For each question, there is a model solution (showing you the level of detail I expect on the exam) and then below that are further comments.

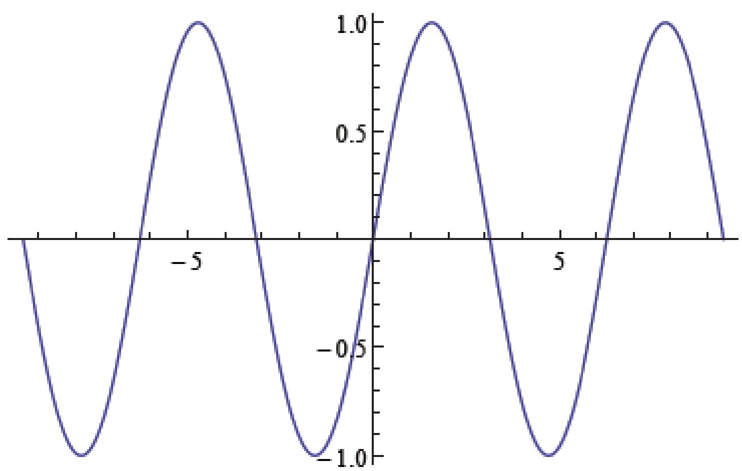
1. *(a) Use scaling and translation to sketch a graph of the following function for $-3\pi \leq x \leq 3\pi$*

$$f(x) = \sin\left(\frac{x}{2} + \pi\right).$$

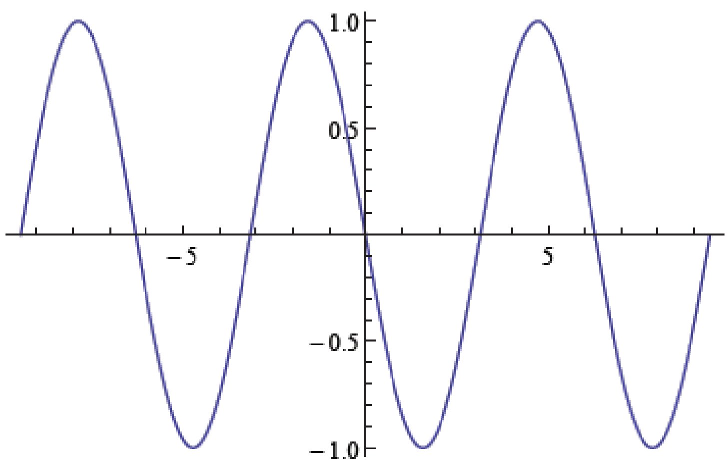
(You should explain how you arrived at your answer. Note that it is not sufficient to create a table of values to plot this graph.)

(b) At what points in $\mathbb{R}$ is this function continuous? (No explanation necessary.)

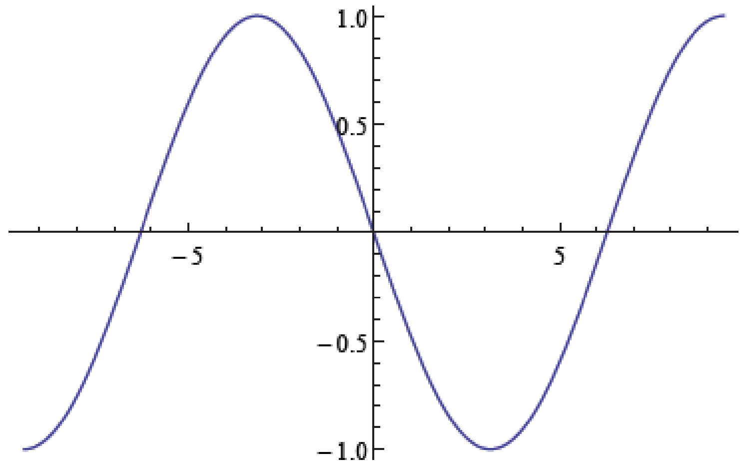
Solution: (a) The graph of $\sin(x)$ is:



Translating by π units to the left we get the graph of $\sin(x + \pi)$:



Stretching by a factor of 2 in the horizontal direction, we get the graph of $\sin\left(\frac{x}{2}+\pi\right)$:



(b) $f(x)$ is continuous at all real numbers.

<u>Comments</u>: It's easy to make the mistake here of first stretching, and then translating, because it seems like the formula $\frac{x}{2}+\pi$ means we should deal with the $\frac{x}{2}$ and then the $+\pi$. But if you do that you get a different graph and plugging in some x-values you see that it is wrong.

There are two ways to think about this. Firstly, when transforming the graph based on changing what is 'inside' the $\sin(-)$, everything is backwards. That is, dividing by 2 *stretches* the graph and adding π moves the graph to the *left*, not the right. The order you do the transformations is also backwards to what you think. First you deal with the $+\pi$ and then the dividing by 2.

Secondly, think about what you do to the formula when you do each step. In particular, stretching horizontally by a factor of 2 corresponds to *replacing x with $\frac{x}{2}$*, and translating left by π units correspond to *replacing x with $x+\pi$*. Therefore, if you first stretch, you get the graph of $\sin(\frac{x}{2})$, but when you then translate, you get the graph of $\sin(\frac{x+\pi}{2})$ which is not what you want.

2. *Calculate the following limit:*

$$\lim_{x\to 2}\frac{\sqrt{x^2+1}-\sqrt{5}}{x-2}.$$

<u>Solution</u>: For $x\neq 2$ we have:

$$\begin{aligned}\frac{\sqrt{x^2+1}-\sqrt{5}}{x-2} &= \frac{(\sqrt{x^2+1}-\sqrt{5})(\sqrt{x^2+1}+\sqrt{5})}{(x-2)(\sqrt{x^2+1}+\sqrt{5})}\\ &= \frac{(x^2-4)}{(x-2)(\sqrt{x^2+1}+\sqrt{5})}\\ &= \frac{x+2}{\sqrt{x^2+1}+\sqrt{5}}\end{aligned}$$

This function is continuous at $x = 2$, so:

$$\begin{aligned}\lim_{x\to 2}\frac{\sqrt{x^2+1}-\sqrt{5}}{x-2} &= \lim_{x\to 2}\frac{x+2}{\sqrt{x^2+1}+\sqrt{5}}\\ &= \frac{2+2}{\sqrt{2^2+1}+\sqrt{5}}\\ &= \frac{4}{2\sqrt{5}}\\ &= \frac{2}{\sqrt{5}}\end{aligned}$$

<u>Comments</u>: This is a standard trick for dealing with square-roots: make sure you know it. Ultimately it relies on the fact that there are two different ways to factor $x^2 - 4$. The more familiar one is

$$x^2 - 4 = (x-2)(x+2)$$

but we also have

$$x^2 - 4 = (x^2+1) - 5 = (\sqrt{x^2+1}-\sqrt{5})(\sqrt{x^2+1}+\sqrt{5}).$$

You can imagine that there are lots of other questions that work the same way.

Also, you *should* say that you are using the fact that the function is continuous when you calculate the limit by substituting in $x = 2$.

3. *Prove, using the precise definition of limit, that:*

$$\lim_{x\to 1}(1-5x) = -4.$$

<u>Solution</u>: <u>Scratch work</u> (you do not *need* to write this but I recommend it):

$$\epsilon > |(1-5x)-(-4)| = |5-5x| = 5|x-1|$$

so $\delta = \epsilon/5$.

<u>Proof</u>: Given $\epsilon > 0$, let $\delta = \epsilon/5 > 0$. Then, if $0 < |x-1| < \delta$, we have

$$5|x-1| < \epsilon$$

and so

$$|5x-5| < \epsilon$$

and so

$$|5-5x| < \epsilon$$

and so

$$|(1-5x)-(-4)| < \epsilon$$

as required.

Comments: The most common error here is to write $|5-5x| = |(-5)(x-1)| = (-5)|x-1|$ and then to try to divide by -5 to get $\delta = \frac{\epsilon}{-5}$. There are two problems with that: (i) δ always has to be positive, so you cannot have $\delta = \epsilon/-5$ as that would be negative (since ϵ is positive), and (ii) it is **not** true that

$$|(-5)(x-1)| = (-5)|x-1|.$$

The left-hand side is positive (since the absolute value of anything is positive) but the right hand-side is negative. What is true instead is that

$$|(-5)(x-1)| = 5|x-1|.$$

(More generally, $|ab| = |a||b|$.) Therefore, the right thing to get is $\delta = \frac{\epsilon}{5}$. This also avoids the problem that if you divide by a negative number, the direction on the inequality would reverse which would also be bad.

4. *Use the definition of derivative to find $f'(2)$ where*

$$f(x) = x^2 + x.$$

Solution: We have

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h\to 0} \frac{[(2+h)^2 + (2+h)] - [2^2+2]}{h} \\ &= \lim_{h\to 0} \frac{4+4h+h^2+2+h-6}{h} \\ &= \lim_{h\to 0} \frac{h^2+5h}{h} \\ &= \lim_{h\to 0}(h+5) \end{aligned}$$

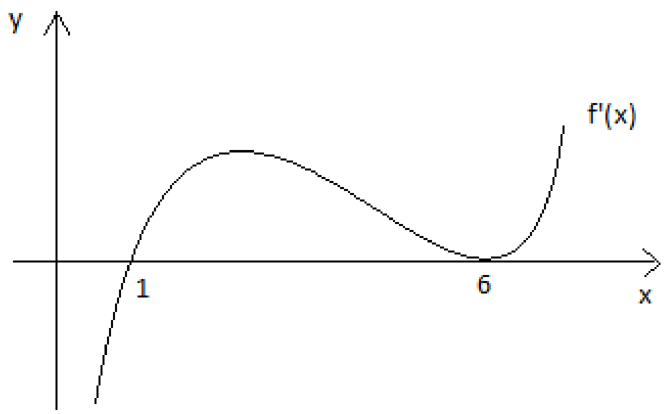
and since $(h+5)$ is continuous, this limit is equal to 5. Therefore $f'(2) = 5$.

Comments: You could also work out $f'(x)$ for general x using the formula

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

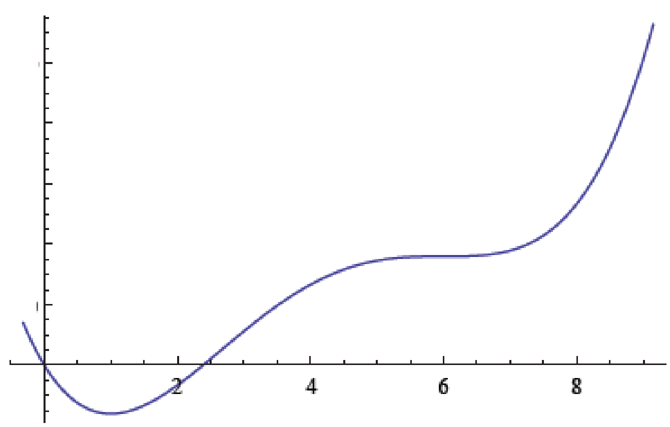
which turns out to be $2x+1$ and then substitute in $x = 2$. But for this test, you are **not** allowed to use the differentiation formulas to find $f'(x)$.

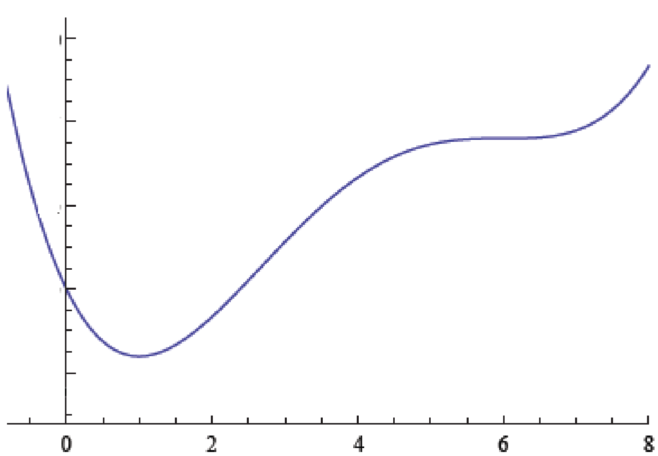
5. *Below is a graph of f' for some function f.*



Sketch two different possible graphs of the function f for the same range of x-values. (Make sure it is clear how your graphs are different, but also how they are related.)

Solution: Two possible graphs of f are





The two graphs are related by the fact that each can be translated vertically by a certain amount to get the other.

Comments: Any graphs of approximately this shape are fine. The key features are that the graph has negative slope for $x < 1$, positive slope for $1 < x < 6$ and for $x > 6$, and that at $x = 1$ and $x = 6$, the tangent line to the graph is horizontal. It might be helpful to you to write out these facts but you don't need to do that **unless the question asks for you to describe how you did it or for specific information like this**. Whenever you are finding the graph of f from that of f' your answer could be shifted up or down any amount to get another answer. This is because, if $g(x) = f(x) + c$ for a constant c, then $g'(x) = f'(x)$. So different functions can have the same derivative. (We'll see later in the semester than the only way this can happen is if they differ by a constant.)