Math 111, Introduction to the Calculus, Fall 2011
Midterm I Solutions

1. *Sketch, on separate diagrams, graphs of the following functions for the range* $0 \le x \le 2\pi$*:*

   (a) $\cos(x)$

   (b) $\cos(2x)$

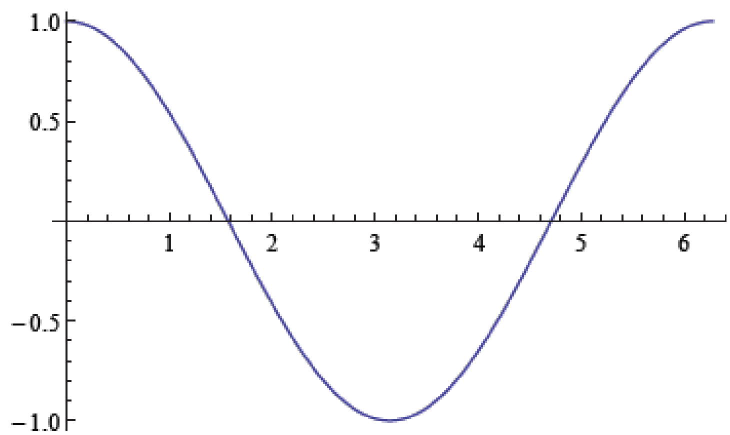
   (c) $\cos\left(x - \frac{\pi}{2}\right)$

   (d) $\cos\left(2x - \frac{\pi}{2}\right)$

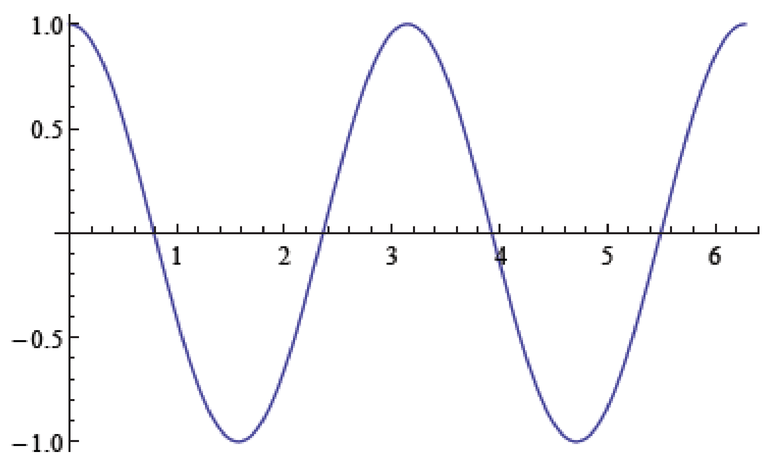
   For parts (b), (c) and (d) explain how you obtained the graph by transforming one of your previous answers. (For each graph, label the x *and* y *axes to make it clear where the graph crosses the* x*-axis, and what the maximum and minimum values are.)*

   Solution:

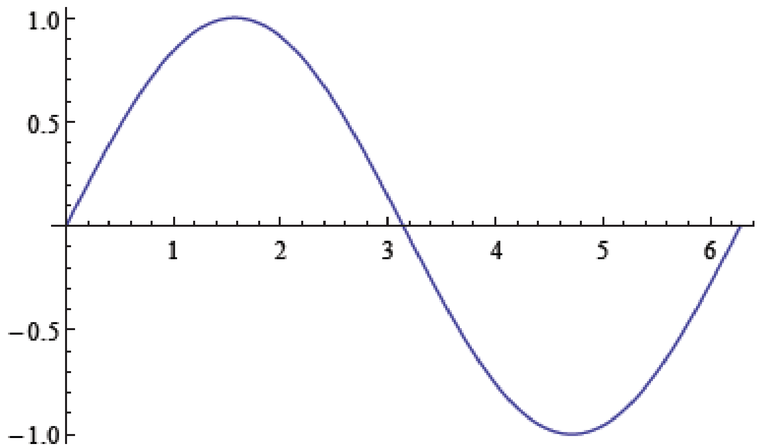
   (a) The graph of $\cos(x)$ is:



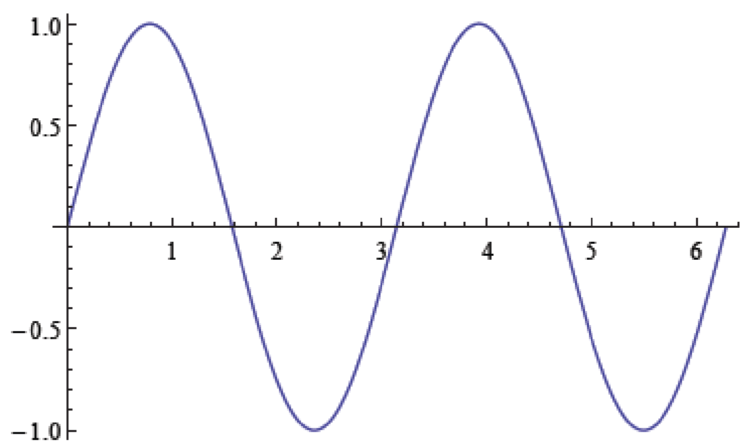
   (b) We obtain this from the graph of $\cos(x)$ by shrinking horizontally by a factor of 2 (since we want to replace x by $2x$ in the formula):



(c) We obtain this from the graph of $\cos(x)$ by translating $\frac{\pi}{2}$ units to the right (since we want to replace x with $x - \frac{\pi}{2}$ in the formula):



(d) We obtain this from the graph of $\cos(x - \frac{\pi}{2})$ by shrinking horizontally by a factor of 2 (since we want to replace x with $2x$ in the formula):



2. *Use the precise definition of a limit to show that*

$$\lim_{x \to 0}(2 - 3x) = 2.$$

<u>Solution</u>: Given $\epsilon > 0$, let $\delta = \epsilon/3$. Then, if $0 < |x - 0| < \delta$, we have

$$|x| < \epsilon/3$$

so

$$3|x| < \epsilon$$

so

$$|3x| < \epsilon$$

so

$$|-3x| < \epsilon$$

so

$$|(2-3x)-2| < \epsilon.$$

3. *Let* $f(x) = \dfrac{1}{1-x}$. *Use the definition of derivative to find* $f'(x)$ *when* $x \neq 1$.

Solution: The definition of derivative gives

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{1-(x+h)} - \frac{1}{1-x}}{h} \\ &= \lim_{h\to 0} \frac{(1-x)-(1-x-h)}{h(1-x-h)(1-x)} \\ &= \lim_{h\to 0} \frac{h}{h(1-x-h)(1-x)} \\ &= \lim_{h\to 0} \frac{1}{(1-x-h)(1-x)} \end{aligned}$$

This function is now continuous at $h=0$ since the denominator is not zero when $h=0$ (since $x \neq 1$). Therefore the limit is equal to

$$\frac{1}{(1-x-0)(1-x)}$$

and so

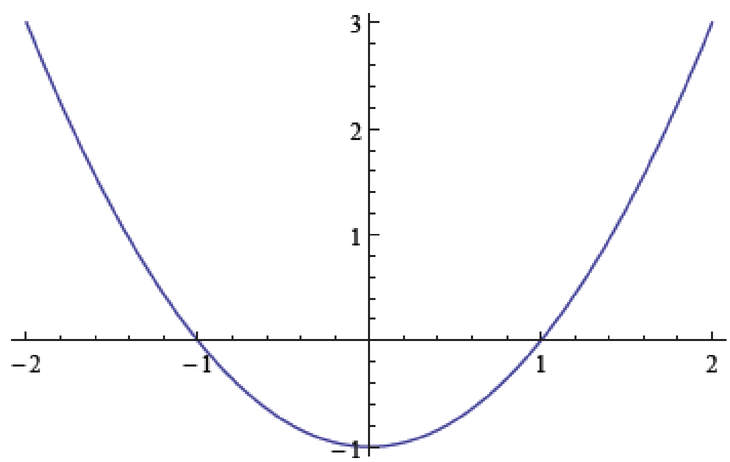
$$f'(x) = \frac{1}{(1-x)^2}.$$

4. (a) *Draw a graph of the function* $f(x) = x^2 - 1$, *marking the points where your graph crosses the* x*-axis.*

   (b) *Use your answer to part (a) to sketch a graph of a possible function* $g(x)$ *such that* $g'(x) = f(x)$. *Explain your answer.*

Solution:

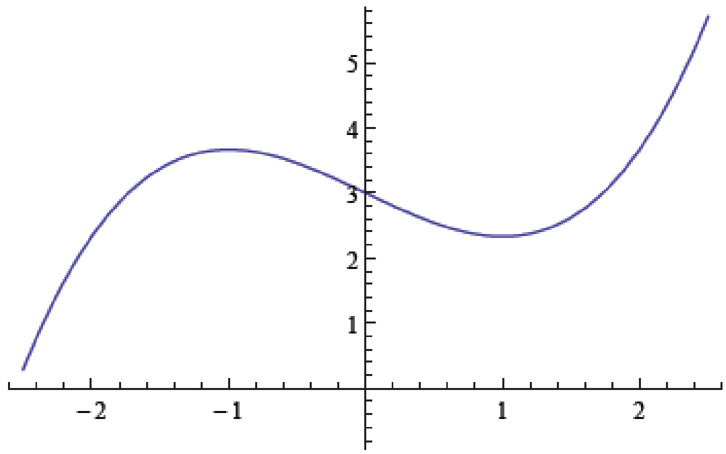
(a) The graph crosses the x-axis at $(-1, 0)$ and $(1, 0)$:



(b) If $g'(x) = f(x)$, then the slope of the graph of $g(x)$ is given by the values of the function in part (a). Therefore, we want the graph of the function $g(x)$ to have

- slope positive for $x < -1$ and $x > 1$
- slope negative for $-1 < x < 1$
- slope zero for $x = -1$ and $x = 1$.

A possible function $g(x)$ is therefore:



5. *Let* $f(x) = |x - 1|$.

   (a) Is $f(x)$ *continuous at* $x = 1$*?*

   (b) Is $f(x)$ *differentiable at* $x = 1$*?*

Justify your answers. (Just drawing graphs is not enough. You should show that the relevant limits either exist or don't, or, are or are not equal to the right thing.)

Solution:

(a) We have to check if

$$\lim_{x \to 1} f(x) = f(1).$$

To find the limit, we calculate the one-sided limits:

$$\begin{aligned} \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} |x - 1| \\ &= \lim_{x \to 1^+} (x - 1) \\ &= (1 - 1) \\ &= 0 \end{aligned}$$

because, when $x > 1$, $x - 1 > 0$ so $|x - 1| = x - 1$. The limit of $(x - 1)$ can be calculated by substituting in $x = 1$ because $x - 1$ is continuous.

We then also have

$$\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} |x - 1| \\ &= \lim_{x \to 1^-} (1 - x) \\ &= (1 - 1) \\ &= 0 \end{aligned}$$

because, when $x < 1$, $x - 1 < 0$, so $|x-1| = -(x-1) = 1-x$. Again, the limit of $1-x$ can be found by substituting in $x = 1$ because $1 - x$ is continuous.

Since the two one-sided limits are equal we now know that

$$\lim_{x\to 1} f(x) = 0.$$

Since $f(1) = |1-1| = 0$ also, this tells us that f *is* continuous at $x = 1$.

(b) We have to check to see if $f'(1)$ exists, that is, if the limit

$$\lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

exists.

This limit is equal to

$$\lim_{h\to 0} \frac{|(1+h)-1| - |1-1|}{h} = \lim_{h\to 0} \frac{|h|}{h}.$$

We again look at the one-sided limits:

$$\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1$$

and

$$\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} -1 = -1.$$

Since the one-sided limits are not equal, we see that

$$\lim_{h\to 0} \frac{|h|}{h}$$

does not exist. Therefore f is *not* differentiable at $x = 1$.