Answer Key, Math 11, Final Examination, December 19, 2009

1. [20 Points] Evaluate each of the following limits. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\lim\limits_{x\to 1} \dfrac{x^2+x-6}{x^2-6x+8} = \boxed{-\dfrac{4}{3}}$ by Direct Substitution

(b) $\lim\limits_{x\to 3} \dfrac{3-x}{|x-3|}$ $\boxed{\text{Does Not Exist}}$ because RHL$\neq$ LHL.

RHL$= \lim\limits_{x\to 3^+} \dfrac{3-x}{x-3} = \lim\limits_{x\to 3^+} \dfrac{-(x-3)}{x-3} = -1$

LHL$= \lim\limits_{x\to 3^-} \dfrac{3-x}{-(x-3)} = \lim\limits_{x\to 3^-} \dfrac{3-x}{3-x} = 1$

(c) $\lim\limits_{x\to 1^+} \dfrac{x^2+x-2}{x^2-2x+1} = \lim\limits_{x\to 1^+} \dfrac{(x+2)(x-1)}{(x-1)(x-1)} = \lim\limits_{x\to 1^+} \dfrac{x+2}{x-1} = \dfrac{3}{0^+} = \boxed{+\infty}$

(d) $\lim\limits_{x\to -7} \dfrac{\dfrac{7}{x}-\dfrac{1}{x+6}}{x+7} = \lim\limits_{x\to -7} \dfrac{\left(\dfrac{7(x+6)-x}{x(x+6)}\right)}{x+7} = \lim\limits_{x\to -7} \dfrac{7(x+6)-x}{x(x+6)(x+7)} = \lim\limits_{x\to -7} \dfrac{7x+42-x}{x(x+6)(x+7)} =$
$\lim\limits_{x\to -7} \dfrac{6x+42}{x(x+6)(x+7)} = \lim\limits_{x\to -7} \dfrac{6(x+7)}{x(x+6)(x+7)} = \lim\limits_{x\to -7} \dfrac{6}{x(x+6)} = \dfrac{6}{-7(-1)} = \boxed{\dfrac{6}{7}}$

2. [30 Points] Compute each of the following derivatives. Do not simplify your answers.

(a) $\dfrac{d}{dx}\left(\dfrac{x^3-\sin(3x)}{e^{4x}}\right) = \boxed{\dfrac{e^{4x}(3x^2-3\cos(3x))-(x^3-\sin(3x))e^{4x}(4)}{e^{8x}}}$

(b) $\dfrac{dy}{dx}$, if $x^2e^y = 1+\ln(xy)$.

First we implicit differentiate both sides w.r.t. x. $\dfrac{d}{dx}\left(x^2e^y\right) = \dfrac{d}{dx}\left(1+\ln(xy)\right)$.

Then $x^2e^y\dfrac{dy}{dx} + e^y2x = \dfrac{1}{xy}\left(x\dfrac{dy}{dx}+y\right) = \dfrac{1}{y}\dfrac{dy}{dx} + \dfrac{1}{x}$

As a result, $\left(x^2e^y - \dfrac{1}{y}\right)\dfrac{dy}{dx} = \dfrac{1}{x} - 2xe^y$

Finally, $\boxed{\dfrac{dy}{dx} = \dfrac{\dfrac{1}{x}-2xe^y}{x^2e^y-\dfrac{1}{y}}}$.

(c) $\dfrac{d}{dx}\left(\int_x^2 \dfrac{\cos t}{3+\sin t}\,dt\right) = -\dfrac{d}{dx}\left(\int_2^x \dfrac{\cos t}{3+\sin t}\,dt\right) = \boxed{-\dfrac{\cos x}{3+\sin x}}$ by FTC Part I

(e) $\dfrac{d}{dx}\left[\ln\left(\dfrac{(x^2+5)^4\,e^{\tan x}}{\sqrt{x^3+2}}\right)\right] = \dfrac{d}{dx}\left[\ln\left((x^2+5)^4\right) + \ln\left(e^{\tan x}\right) - \ln\sqrt{x^3+2}\right]$

$= \dfrac{d}{dx}\left[4\ln(x^2+5) + \tan x - \ln\left((x^3+2)^{\frac{1}{2}}\right)\right] = \dfrac{d}{dx}\left[4\ln(x^2+5) + \tan x - \dfrac{1}{2}\ln(x^3+2)\right]$

$= \dfrac{4}{x^2+5}(2x) + \sec^2 x - \dfrac{1}{2}\dfrac{1}{x^3+2}(3x^2) = \boxed{\dfrac{8x}{x^2+5} + \sec^2 x - \dfrac{3x^2}{2(x^3+2)}}$

(f) $f''(x)$, where $f(x) = e^{\sin x} + \ln\sqrt{x}$.

First, $f'(x) = e^{\sin x}\cos x + \dfrac{1}{\sqrt{x}}\dfrac{1}{2\sqrt{x}} = e^{\sin x}\cos x + \dfrac{1}{2x}$

Second, $f''(x) = e^{\sin x}(-\sin x) + \cos x\left(e^{\sin x}\right)\cos x - \dfrac{1}{2x^2} = \boxed{-\sin x e^{\sin x} + \cos^2 x\left(e^{\sin x}\right) - \dfrac{1}{2x^2}}$

(g) $f'(x)$, where $f(x) = x^{\sin x}$.

We can solve this two ways: first try Logarithmic Differentiation and using the properties of logs,

Let $y = x^{\sin x}$, so that $\ln y = \ln(x^{\sin x}) = \sin x \ln x$

Next use implicit differentation to differentiate both sides w.r.t x.

$\dfrac{d}{dx}(\ln y) = \dfrac{d}{dx}(\sin x \ln x)$

Then $\dfrac{1}{y}\dfrac{dy}{dx} = \sin x\left(\dfrac{1}{x}\right) + \ln x \cos x$.

As a result, $\dfrac{dy}{dx} = y\left(\dfrac{\sin x}{x} + \ln x\cos x\right)$.

Finally, $\dfrac{dy}{dx} = \boxed{x^{\sin x}\left(\dfrac{\sin x}{x} + \ln x\cos x\right)}$.

The second option is to rewrite $y = x^{\sin x} = e^{\ln\left(x^{\sin x}\right)} = e^{\sin x \ln x}$.

Then differentiate, $\dfrac{d}{dx}\left(x^{\sin x}\right) = \dfrac{d}{dx}\left(e^{\sin x\ln x}\right) = e^{\sin x \ln x}\left(\sin x\left(\dfrac{1}{x}\right) + \ln x\cos x\right)$

$= e^{\ln\left(x^{\sin x}\right)}\left(\dfrac{\sin x}{x} + \ln x\cos x\right) = \boxed{x^{\sin x}\left(\dfrac{\sin x}{x} + \ln x\cos x\right)}$.

3. [25 Points] Compute each of the following integrals.

(a) $\displaystyle\int \frac{(1+\sqrt{x})^2}{x\sqrt{x}}\,dx = \int \frac{1+2\sqrt{x}+x}{x\sqrt{x}}\,dx = \int \frac{1}{x^{\frac{3}{2}}} + \frac{2}{x} + \frac{1}{\sqrt{x}}\,dx = \int x^{-\frac{3}{2}} + \frac{2}{x} + x^{-\frac{1}{2}}\,dx$

$$= -2x^{-\frac{1}{2}} + 2\ln|x| + 2x^{\frac{1}{2}} + C = \boxed{-\frac{2}{\sqrt{x}} + 2\ln|x| + 2\sqrt{x} + C}$$

(b) $\displaystyle\int_0^3 |x-1|\ dx = \int_0^1 -(x-1)\ dx + \int_1^3 x - 1\ dx = x - \frac{x^2}{2}\Big|_0^1 + \frac{x^2}{2} - x\Big|_1^3$

$$= \left(1 - \frac{1}{2}\right) - 0 + \left(\frac{9}{2} - 3\right) - \left(\frac{1}{2} - 1\right) = \frac{1}{2} + \frac{9}{2} - 3 + \frac{1}{2} = \frac{9}{2} - 2 = \frac{9}{2} - \frac{4}{2} = \boxed{\frac{5}{2}}$$

(c) $\displaystyle\int \frac{e^x + \cos x}{e^x + \sin x}\ dx = \int \frac{1}{u}\ du = \ln|u| + C = \boxed{\ln|e^x + \sin x| + C}$

Here $\begin{cases} u &= e^x + \sin x \\ du &= e^x + \cos x dx \end{cases}$

(d) $\displaystyle\int e^{6x}\sqrt{7 + e^{6x}}\ dx = \frac{1}{6}\int \sqrt{u}\ du = \frac{1}{6}\left(\frac{2}{3}\right)u^{\frac{3}{2}} + C = \boxed{\frac{1}{9}\left(7 + e^{6x}\right)^{\frac{3}{2}} + C}$

Here $\begin{cases} u &= 7 + e^{6x} \\ du &= 6e^{6x}dx \\ \frac{1}{6}du &= e^{6x}dx \end{cases}$

(e) $\displaystyle\int_e^{e^4} \frac{1}{x\sqrt{\ln x}}\ dx = \int_{u=1}^{u=4} \frac{1}{\sqrt{u}}\ du = 2\sqrt{u}\Big|_{u=1}^{u=4} = 2\sqrt{4} - 2\sqrt{1} = 4 - 2 = \boxed{2}$

Here $\begin{cases} u &= \ln x \\ du &= \frac{1}{x}dx \end{cases}$ and $\begin{cases} x = e & \Longrightarrow u = \ln e = 1 \\ x = e^4 & \Longrightarrow u = \ln e^4 = 4 \end{cases}$

4. [10 Points] Give an ε-δ proof that $\displaystyle\lim_{x\to 1} 8 - 3x = 5$.

Scratchwork: we want $|f(x) - L| = |(8 - 3x) - 5| < \varepsilon$

$$|f(x) - L| = |(8 - 3x) - 5| = |3 - 3x| = |-3(x-1)| = |-3||x-1| = 3|x-1| \text{ (want } < \varepsilon)$$

$$3|x-1| < \varepsilon \text{ means } |x-1| < \frac{\varepsilon}{3}$$

$$\text{So choose } \delta = \frac{\varepsilon}{3} \text{ to restrict } 0 < |x-1| < \delta. \text{ That is } 0 < |x-1| < \frac{\varepsilon}{3}.$$

Proof: Let $\varepsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{3}$. Given x such that $0 < |x-1| < \delta$, then

$$|f(x) - L| = |(8 - 3x) - 5| = |3 - 3x| = |-3(x-1)| = |3||x-1| = 3|x-1| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon.$$

$\square$

5. [10 Points] Let $f(x) = \sqrt{7x-3}$. Calculate $f'(x)$, using the **limit definition** of the derivative.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{7(x+h)-3}-\sqrt{7x-3}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{7(x+h)-3}-\sqrt{7x-3}}{h} \cdot \frac{\sqrt{7(x+h)-3}+\sqrt{7x-3}}{\sqrt{7(x+h)-3}+\sqrt{7x-3}} = \lim_{h\to 0} \frac{(7(x+h)-3)-(7x-3)}{h(\sqrt{7(x+h)-3}+\sqrt{7x-3})}$$

$$= \lim_{h\to 0} \frac{7x+7h-3-7x+3}{h(\sqrt{7(x+h)-3}+\sqrt{7x-3})} = \lim_{h\to 0} \frac{7h}{h(\sqrt{7(x+h)-3}+\sqrt{7x-3})}$$

$$= \lim_{h\to 0} \frac{7}{\sqrt{7(x+h)-3}+\sqrt{7x-3}} = \boxed{\frac{7}{2\sqrt{7x-3}}}$$

6. [10 Points] Suppose that $f(x) = \ln 2 + \ln(\cos x)$. Write the **equation** of the tangent line to the curve $y = f(x)$ when $x = \frac{\pi}{3}$.

First the slope $f'(x) = \frac{1}{\cos x}(-\sin x)$. Then $f'\left(\frac{\pi}{3}\right) = \frac{-\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$.

Then the y-value is $f\left(\frac{\pi}{3}\right) = \ln 2 + \ln\left(\cos\left(\frac{\pi}{3}\right)\right) = \ln 2 + \ln\left(\frac{1}{2}\right) = \ln 2 + \ln 1 - \ln 2 = \ln 1 = 0$

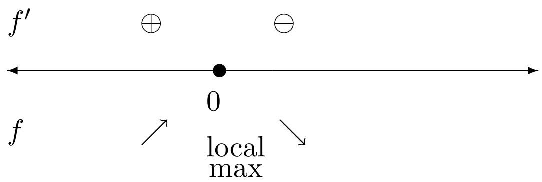
Therefore, the equation of the tangent line through the point $\left(\frac{\pi}{3}, 0\right)$ with slope $-\sqrt{3}$, is $y - 0 = -\sqrt{3}(x - \frac{\pi}{3})$ or $\boxed{y = -\sqrt{3}x + \frac{\sqrt{3}\pi}{3}}$.

7. [20 Points] Sketch the graph of $y = f(x)$, where $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

This function is called the standard normal distribution, which is one of the most important models in statistics. Discuss any symmetry present in the graph. Clearly indicate horizontal asymptote(s), local minima/maxima, and inflection point(s) on the graph, as well as where the graph is increasing, decreasing, concave up and concave down. Take my word for it that

$$f'(x) = -\frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad \text{and} \qquad f''(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}(x^2-1).$$

- $f(x)$ has domain $(-\infty, \infty)$ so No Vertical Asymptotes, even though we didn't ask for them.
- Symmetry: $f(x)$ is an **even** function since $f(-x) = f(x) \Longrightarrow$ symmetry about y-axis.
- There are horizontal asymptotes for this f at $y = 0$ since $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$.
- First Derivative Information: The critical points occur where f' is undefined (never here) or zero ($x = 0$). As a result, $x = 0$ is the critical number. Using sign testing/analysis for f',

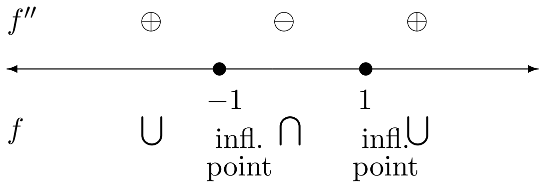


Therefore, f is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$ with a local max at $(0, f(0)) = (0, \frac{1}{\sqrt{2\pi}})$.

• Second Derivative Information

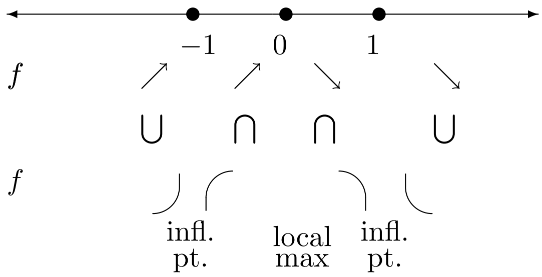
Possible inflection points occur when f'' is undefined (never here) or zero ($x = \pm 1$).

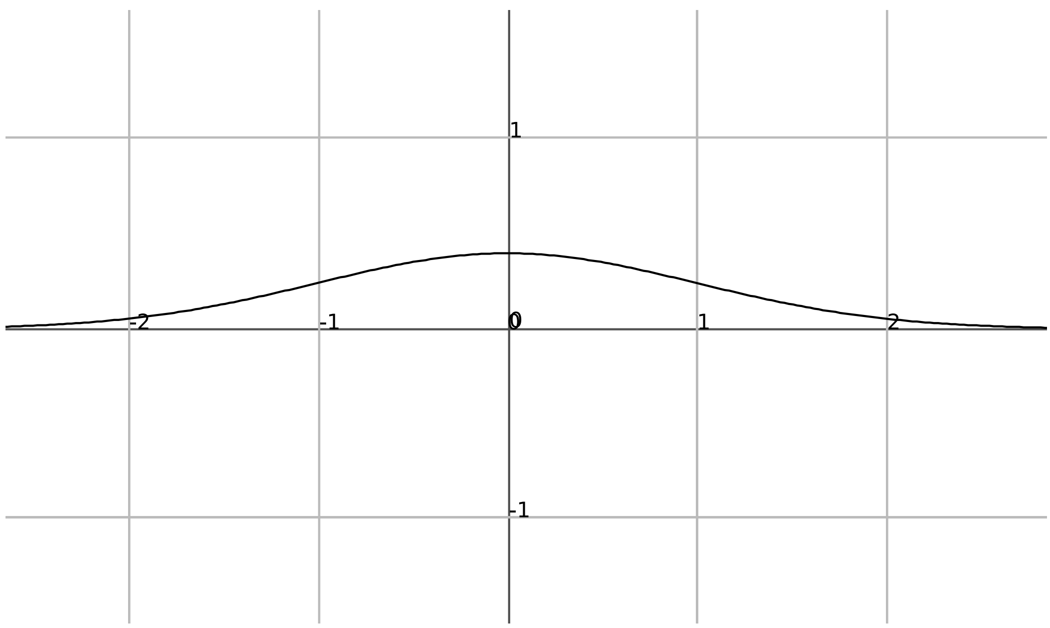
Using sign testing/analysis for f'',



Therefore, f is concave up on $(-\infty, -1)$ and $(1, \infty)$, whereas f is concave down on $(-1, 1)$ with I.P. at $(\pm 1, \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}})$.

• Piece the first and second derivative information together

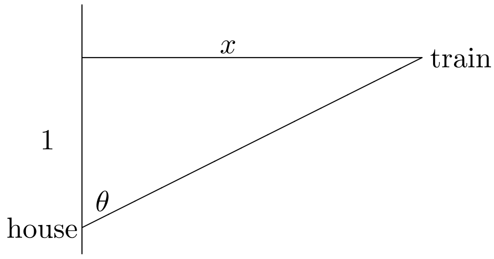




8. [15 Points] A train is travelling east on a straight track at 40 mph. The track is crossed by a road going north and south, and a house is on the road one mile south of the track. Draw the straight line connecting the house to the train. Consider the angle between the road and this line, as measured at the house. How fast is this angle changing when the train is 3 miles east of the road?

(a) • Diagram

The picture at arbitrary time t is:



• Variables

Let $x =$ distance train has travelled horizontally(east) at time t

Find $\dfrac{d\theta}{dt} =?$ when $x = 3$ feet

and $\dfrac{dx}{dt} = 40\dfrac{\text{mi}}{\text{hr}}$

• Equation relating the variables:

The trigonometry of the triangle yields $\tan\theta = \dfrac{x}{1}$

• Differentiate both sides w.r.t. time t.

$\dfrac{d}{dt}(\tan\theta) = \dfrac{d}{dt}(x) \implies \sec^2\theta\dfrac{d\theta}{dt} = \dfrac{dx}{dt}$

• Substitute Key Moment Information (now and not before now!!!):

At the key instant when $x = 3$, using the trigonometry of the triangle, we have a $1, 3, \sqrt{10}$ triangle snap shot. Then $\sec\theta = \dfrac{\text{hypotenuse}}{\text{adjacent}} = \dfrac{\sqrt{10}}{1}$

So, $\sec^2\theta\dfrac{d\theta}{dt} = \dfrac{dx}{dt}$

becomes

$(\sqrt{10})^2\dfrac{d\theta}{dt} = 40$

• Solve for the desired quantity:

$\dfrac{d\theta}{dt} = \dfrac{40}{10} = 4\dfrac{\text{rad}}{\text{hr}}$

• Answer the question that was asked: The angle is growing at 4 radians every hour.

9. [20 Points] Let R be the region between $y = 9 - x^2$ and the x-axis. Find the area of the largest rectangle that can be inscribed in the region R. Two vertices of the rectangle lie on the x-axis. Its other two vertices above the x-axis lie on the parabola $y = 9 - x^2$.

(Remember to state the domain of the function you are computing extreme values for.)

Draw a diagram.

Then the area $A = 2xy = 2x(9 - x^2) = 18x - 2x^3$ must be maximized.

The (common-sense-bounds)domain of A is $\{x : 0 \le x \le 3\}$.

Next $A' = 18 - 6x^2$. Setting $A' = 0$ we solve for $x^2 = 3$ or $x = \sqrt{3}$. (We take the positive square root here because we're talking distance.)

Sign-testing the critical number does indeed yield a maximum for the area function.

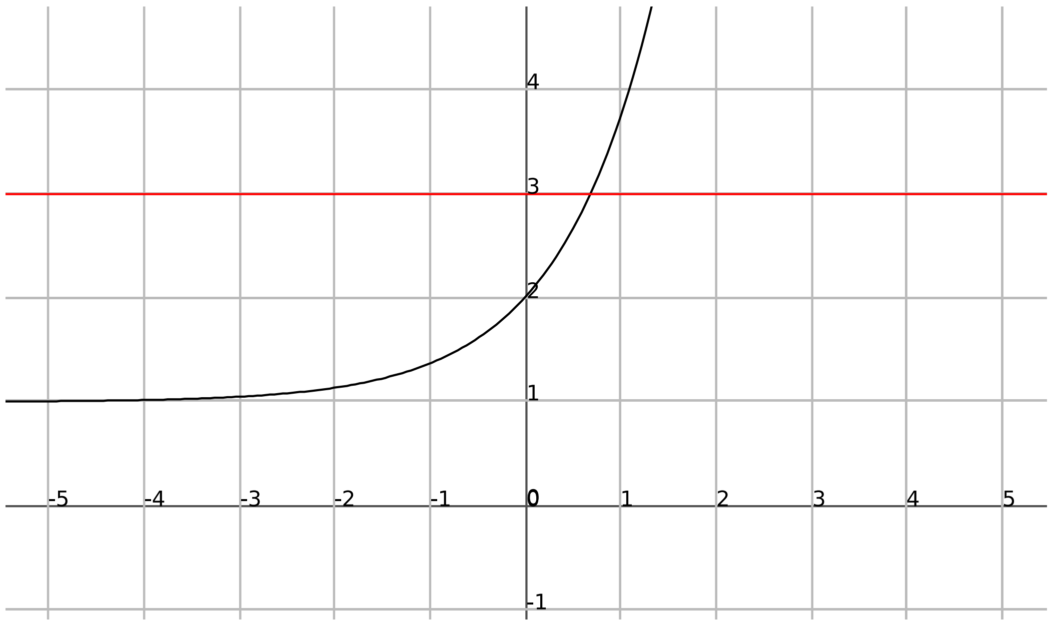
$$\begin{array}{c|cc} A' & \oplus & \ominus \\ \hline A & \nearrow \sqrt{3} & \searrow \end{array}$$

MAX

Since $x = \sqrt{3}$ then $y = 9 - (\sqrt{3})^2 = 6$. As a result, the largest area that occurs is $A = 2xy = 2\sqrt{3}(6) = 12\sqrt{3}$ square units.

10. [20 Points] Consider the region in the plane bounded by the curves $x = 0$, $y = 3$ and $y = 1 + e^x$.

(a) Draw a picture of the region.



Note that the curves intersect when $1 + e^x = 3$ which is when $e^x = 2$ which implies $x = \ln 2$.

(b) Compute the area of the region.

$$\text{Area} = \int_0^{\ln 2} \text{top} - \text{bottom}\ dx = \int_0^{\ln 2} 3 - (1 + e^x)\ dx = \int_0^{\ln 2} 2 - e^x\ dx = 2x - e^x \Big|_0^{\ln 2}$$

$$= (2\ln 2 - e^{\ln 2}) - (0 - e^0) = 2\ln 2 - 2 + 1 = \boxed{\ln 4 - 1}$$

(c) Compute the volume of the 3-dimensional object obtained by rotating the region about the x-axis.

$$\text{Volume} = \int_0^{\ln 2} \pi[(\text{outer radius})^2 - (\text{inner radius})^2]\ dx = \int_0^{\ln 2} \pi[3^2 - (1 + e^x)^2]\ dx$$

$$= \int_0^{\ln 2} \pi[9 - (1 + 2e^x + e^{2x})]\ dx = \int_0^{\ln 2} \pi[8 - 2e^x - e^{2x}]\ dx = \pi[8x - 2e^x - \frac{1}{2}e^{2x}]\Big|_0^{\ln 2}$$

$= \pi[8\ln 2 - 2e^{\ln 2} - \frac{1}{2}e^{2\ln 2} - (0 - 2e^0 - \frac{1}{2}e^0)] = \pi[8\ln 2 - 2(2) - \frac{1}{2}e^{\ln 4} + 2 + \frac{1}{2}] = \pi[8\ln 2 - 2(2) - \frac{1}{2}(4) + 2 + \frac{1}{2}] = \pi[8\ln 2 - 2(2) - 2 + 2 + \frac{1}{2}] = \pi[8\ln 2 - 4 + \frac{1}{2}] = \boxed{\pi[8\ln 2 - \frac{7}{2}]}$

11. [20 Points] Consider an object moving on the number line such that its velocity at time t is $v(t) = \sin(t) + 1$ ft/sec. Also assume that $s(0) = 3$ ft, where as usual $s(t)$ is the position of the object at time t.

(a) Compute the acceleration $a(t)$ and position $s(t)$.

$\boxed{a(t) = \cos t}$

$s(t) = -\cos t + t + C$, and we use the initial condition $s(0) = 3$ now.

$s(0) = -\cos 0 + 0 + C = 3$. Solve this equation for the constant of integration $C = 4$. As a result, $\boxed{s(t) = -\cos t + t + 4}$.

(b) Draw the graph of $v(t)$ for $0 \leq t \leq 2\pi$ and explain why the object is always moving to the right.

I don't have a graph here, but it's just the $\sin t$ function shifted up by 1.

Note that $v(t)$ is always non-negative so the function is never decreasing (or moving to the left). Although the velocity equals zero at one point, it is never stopped for any length of time. That is the position function is always strictly increasing, so it is moving to the right.

(c) Compute the total distance travelled for $\pi/2 \leq t \leq 2\pi$.

Total Distance= $\int_{\frac{\pi}{2}}^{2\pi} |v(t)|\ dt$. Note again that the velocity function is already greater than or equal to zero, so we don't need the absolute value piece.

Therefore, Total Distance= $\int_{\frac{\pi}{2}}^{2\pi} |v(t)|\ dt = \int_{\frac{\pi}{2}}^{2\pi} |\sin t + 1|\ dt = \int_{\frac{\pi}{2}}^{2\pi} \sin t + 1\ dt = -\cos t + t\Big|_{\frac{\pi}{2}}^{2\pi} =$
$(-\cos 2\pi + 2\pi) - (-\cos\frac{\pi}{2} + \frac{\pi}{2}) = -\cos 2\pi + 2\pi + \cos\frac{\pi}{2} - \frac{\pi}{2} = -1 + 2\pi + 0 - \frac{\pi}{2} = \boxed{-1 + \frac{3\pi}{2}}$.