Elementary Vector Analysis: Preliminaries
(from Stewart, *Calculus*, Chapters 12 and 13)

Three-Dimensional Coordinate Systems

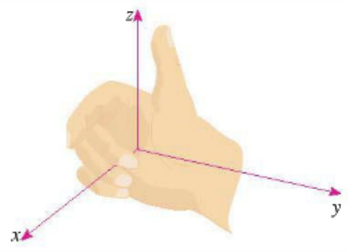
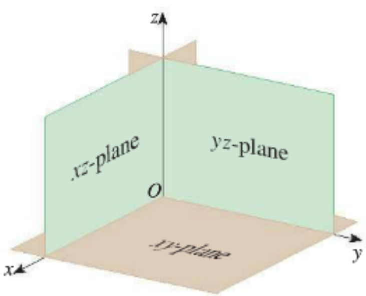


FIGURE 2
Right-hand rule

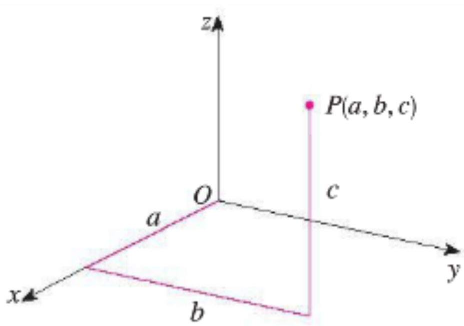
Right hand rule: to choose an orientation on $\mathbb{R}^3 = \mathbb{R}\times\mathbb{R}\times\mathbb{R}$

Curl fingers around the z-axis in the direction off a counter-clockwise rotation from the positive x-axis to the positive y-axis. The thumb points up in the direction of the positive z-axis.



Coordinate planes

The first octant is determined by the positive axes.



How to plot a point

Move a units along the x-axis
then move b units parallel to the y-axis
then c units parallel to the z-axis.

The Distance Formula

Two dimensions: The distance between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Three dimensions: The distance between two points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Vectors

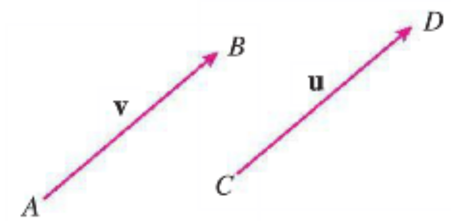


FIGURE 1
Equivalent vectors

A vector is a quantity that has both a magnitude and a direction.

$\vec{v} = \vec{AB}$ has initial point A and terminal point B.

Two vectors with the same length and direction are equivalent, regardless of their position in space, i.e. $\vec{v} = \vec{u}$.

The zero vector $\vec{0}$ has length 0 and no specific direction.

Vector Addition

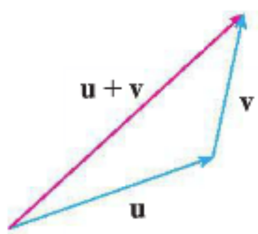


FIGURE 3 The Triangle Law

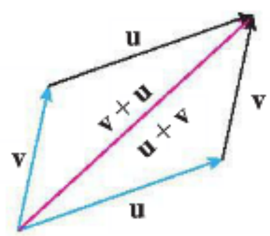
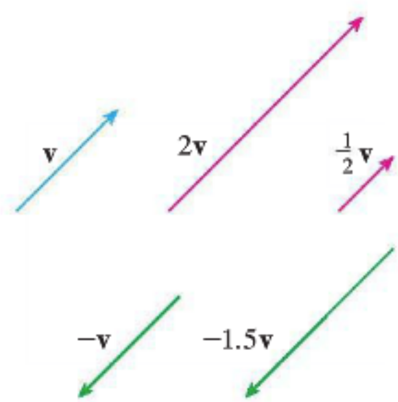


FIGURE 4 The Parallelogram Law

Scalar multiplication

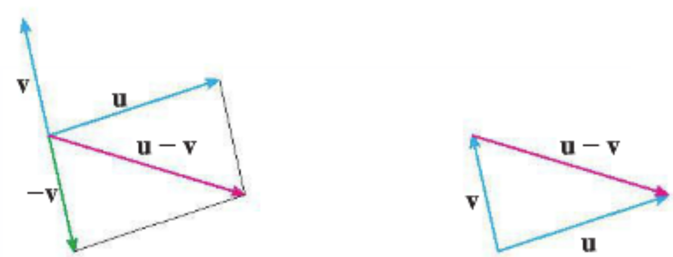
If $c \in \mathbb{R}$ is a scalar and $\vec{v}$ is a vector, the scalar multiple $c\vec{v}$ is the vector whose length is $|c|\|\vec{v}\|$ and whose direction is the same as $\vec{v}$ if $c > 0$ and is opposite to $\vec{v}$ if $c < 0$. If $c = 0$ or $\vec{v} = 0$ then $c\vec{v} = \vec{0}$.

Two nonzero vectors are parallel if and only if they are scalar multiples of each other.



Vector subtraction

$\vec{u} - \vec{v} = \vec{u} + -\vec{v}$



(a) (b)

Vector Components

If a vector $\vec{a}$ has its initial point at the origin $(0,0,0)$ and its terminal point at $P = (a_1, a_2, a_3)$, we write $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and call $\vec{a}$ the position vector of the point P.

Length or Magnitude of a vector is the distance between its initial and terminal points.

Two dimensions: $\|\vec{a}\| = \sqrt{a_1^2 + a_2^2}$

Three dimensions: $\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

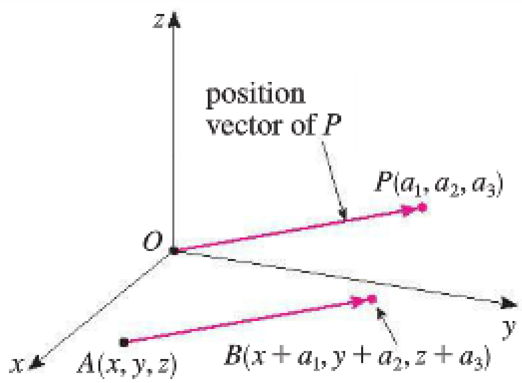


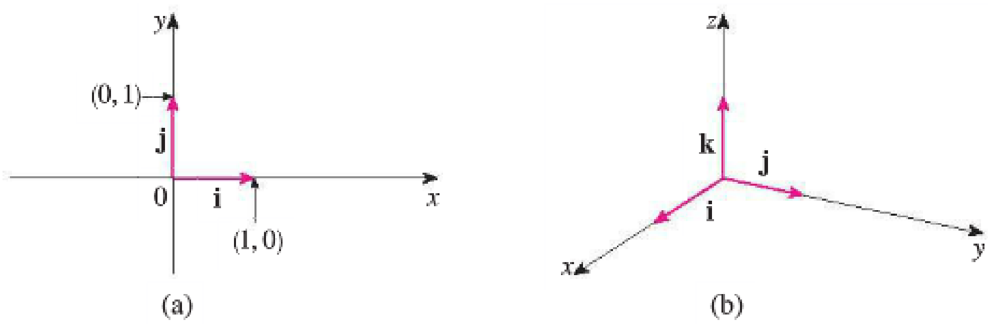
FIGURE 13
Representations of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

A unit vector is a vector with length one. The vector $\vec{u} = \frac{\vec{a}}{\|\vec{a}\|}$ is a unit vector in the direction of the vector $\vec{a}$.

Vector Algebra: Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_2 \rangle$ and $c \in \mathbb{R}$. Then

(1) $\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$
(2) $\vec{a} - \vec{b} = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$
(3) $c\vec{a} = \langle ca_1, ca_2, ca_3 \rangle$

Standard Basis Vectors: $\langle a_1, a_2, a_3 \rangle = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$



$\vec{i} = \langle 1, 0 \rangle$, $\vec{j} = \langle 0, 1 \rangle$

$\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$, $\vec{k} = \langle 0, 0, 1 \rangle$

Properties of Vectors

If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors and s and t are scalars, then

- $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
- $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$
- $\vec{a} + \vec{0} = \vec{a}$
- $\vec{a} + (-\vec{a}) = \vec{0}$
- s$(\vec{a} + \vec{b}) = s\vec{a} + s\vec{b}$
- $(s + t)\vec{a} = s\vec{a} + t\vec{a}$
- $(st)\vec{a} = s(t\vec{a})$
- $1 \cdot \vec{a} = \vec{a}$

The Dot Product: $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

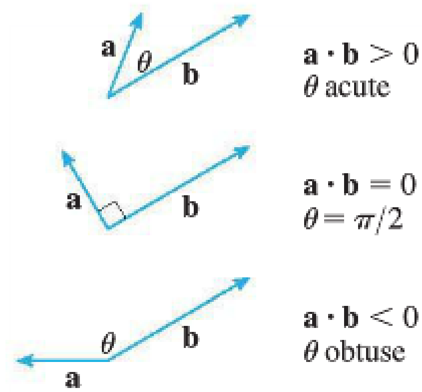
Properties of the Dot Product: If $\vec{a}$, $\vec{b}$, $\vec{c}$ are vectors and s is a scalar, then

(1) $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$
(2) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
(3) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
(4) $(s\vec{a}) \cdot \vec{b} = s(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (s\vec{b})$
(5) $\vec{0} \cdot \vec{a} = \vec{0}$

Theorem. Let $0 \leq \theta \leq \pi$ be the angle between the vectors $\vec{a}$ and $\vec{b}$. Then

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta.$$

Corollary. Two nonzero vectors $\vec{a}$ and $\vec{b}$ are orthogonal (perpendicular) if and only if $\vec{a} \cdot \vec{b} = 0$.



The Cross Product

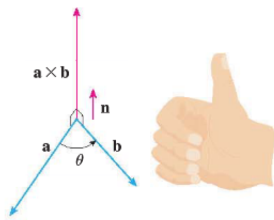
Recall the determinant of a 2×2 matrix is $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

The cross product of two vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ in $\mathbb{R}^3$ is

$$\begin{aligned} \vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k} \\ &= \langle a_2 b_3 - a_3 b_2, -(a_1 b_3 - a_3 b_1), a_1 b_2 - a_2 b_1 \rangle \end{aligned}$$

Geometric Interpretations of the Cross Product

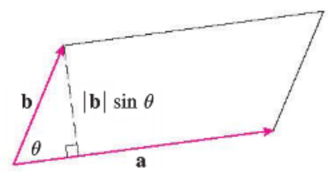
(1) $\vec{a} \times \vec{b}$ is a vector orthogonal to both $\vec{a}$ and $\vec{b}$, pointing in the direction determined by the



right hand rule.

(2) $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin(\theta)$ where $\theta \in [0, \pi]$ is the angle between $\vec{a}$ and $\vec{b}$.

(3) $\|\vec{a} \times \vec{b}\|$ is the area of the parallelogram with sides $\vec{a}$ and $\vec{b}$.



Corollary. Two nonzero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a} \times \vec{b} = \vec{0}$.

Algebraic Properties of Cross Product: Let $\vec{a}, \vec{b}, \vec{c} \in V_3$ and $s \in \mathbb{R}$. Then

(1) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
(2) $(s\vec{a}) \times \vec{b} = s(\vec{a} \times \vec{b}) = \vec{a} \times (s\vec{b})$
(3) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$
(4) $(\vec{a} + \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c})$
(5) $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$