1. Elementary Vector Analysis

(from Stewart, *Calculus*, Chapters 12 and 13)

Vectors, Points, and Position Vectors: The position vector of a point P is the vector from the origin to P. For example, in $\mathbb{R}^3$, a vector $\vec{v} = \langle x, y, z \rangle$ is the position vector of the point $P = (x, y, z)$. In this way, we often identify points as vectors and vice versa.

Parametrized Curves

Parametrized Curves: Given a continuous vector function $\vec{r}(t) = \langle x(t), y(t) \rangle$ or $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, the set of points with position vector $\vec{r}(t)$ for some t defines a curve C. The components $x(t), y(t)$, and $z(t)$ are called parametric equations for the parametrized curve C. The variable t is called the parameter.

The tangent vector to a curve at the point $P = \vec{r}(t)$ is $\vec{r'}(t) = \langle x'(t), y'(t) \rangle$ or $\vec{r'}(t) = \langle x'(t), y'(t), z'(t) \rangle$. The line through the point $P = \vec{r}(t)$ parallel to the vector $\vec{r'}(t)$ is called the tangent line to C at P.

Ex. 1. (a) Sketch the curve C with parametric equations $x(t) = t + 1$, $y(t) = t^2 - 2t$, $-1 \leq t \leq 3$.
(b) Show that C is an arc of the parabola $y = x^2 - 4x + 3$ and use this fact to give another parametrization of C.

t	x	y
-1	0	3
0	1	0
1		
2		
3		

Ex. 2. What curve is represented by the parametric equations?

(a) $x(t) = 2\cos(t)$, $y(t) = 2\sin(t)$, $0 \leq t \leq 2\pi$

(b) $x(t) = \sin(2t)$, $y(t) = \cos(2t)$, $0 \leq t \leq 2\pi$

(c) Compute the tangent vector to the curve $\vec{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle$ and show that it is perpendicular to the curve at any point $(x(t), y(t))$.

Lines in $\mathbb{R}^2$ and $\mathbb{R}^3$

Vector Equation: $\vec{r}(t) = \vec{r_0} + t\vec{v}$

Parametric Equations: $x(t) = x_0 + at,\ y(t) = y_0 + bt,\ z(t) = z_0 + ct$

$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ are the position vectors of the points $(x(t), y(t), z(t))$ on the line
$\vec{r_0} = \langle x_0, y_0, z_0\rangle$ is the position vector of a fixed point $P_0 = (x_0, y_0, z_0)$ on the line
$\vec{v} = \langle a, b, c\rangle$ is a direction vector of the line, i.e. $\vec{v}$ is any vector parallel to the line

Rmk. For lines in $\mathbb{R}^2$, just delete all of the third coordinates above.

Ex. 3. Find parametric equations for the line that passes through the points $(2, 0, 1)$ and $(-1, 1, -1)$.

Ex. 4. Find parametric equations for the line segment from $(3, 1, 2)$ to $(-5, 4, 1)$.

Ex. 5. Find parametric equations for the line that passes through the point $(2, 0, 1)$ and is parallel to the line given by parametric equations $x(t) = 3t$, $y(t) = -5$, and $z(t) = t + 4$.

Ex. 6. Find parametric equations for the tangent line to the parametrized curve $x(t) = t+1$, $y(t) = t^2 - 2t$, at the point $(0, 3)$.

Planes in $\mathbb{R}^3$

Vector Equation: $\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$

Scalar Equations:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

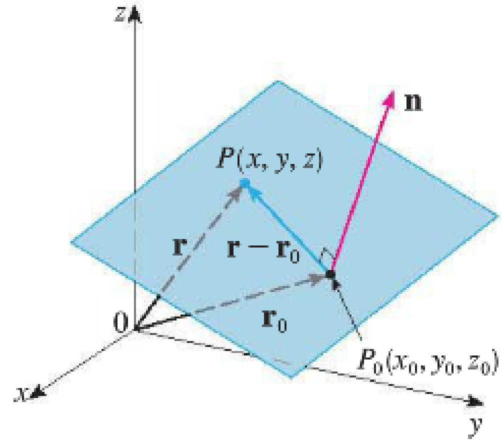
or

$$ax + by + cz = d$$

$\vec{r} = \langle x, y, z \rangle$ are the position vectors of points (x, y, z) in the plane

$\vec{r_0} = \langle x_0, y_0, z_0 \rangle$ is the position vector of a fixed point $P_0 = (x_0, y_0, z_0)$ in the plane

$\vec{n} = \langle a, b, c \rangle$ is the normal vector, i.e. any vector orthogonal to the plane



Ex. 7. Find an equation of the plane that passes through the point $(1, 1, 0)$ and is perpendicular to the line with parametric equations $x(t) = 2 + 3t$, $y(t) = 1 - t$, $z(t) = 2 + 2t$.

Ex. 8. Find an equation of the plane that passes through the points $(1, 0, 2)$, $(4, 2, 3)$, and $(2, 0, 4)$.

Ex. 9. Find an equation of the plane that passes through the point $(1, 0, 2)$ and contains the line

$$x = 2t + 2, y = 2t, z = 4 - t.$$

Ex. 10. Find parametric equations for the line of intersection for the two planes $2x + y + z = 8$ and $x + 2y - z = 1$.