1. Vector Spaces and Subspaces

(References: Comps Study Guide for Linear Algebra Section 1;
Damiano & Little, *A Course in Linear Algebra*, Chapter 1)

Vector Spaces

(1.1.1) Definition. A (real) *vector space* is a set V (whose elements are called *vectors* by analogy with the first example we considered) together with

a) an operation called *vector addition*, which for each pair of vectors $\mathbf{x}$, $\mathbf{y} \in V$ produces another vector in V denoted $\mathbf{x} + \mathbf{y}$, and

b) an operation called *multiplication by a scalar* (a real number), which for each vector $\mathbf{x} \in V$, and each scalar $c \in \mathbf{R}$ produces another vector in V denoted $c\mathbf{x}$.

Furthermore, the two operations must satisfy the following *axioms:*

1. For all vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z} \in V$, $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.
2. For all vectors $\mathbf{x}$ and $\mathbf{y} \in V$, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.
3. There exists a vector $\mathbf{0} \in V$ with the property that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for all vectors $\mathbf{x} \in V$.
4. For each vector $\mathbf{x} \in V$, there exists a vector denoted $-\mathbf{x}$ with the property that $\mathbf{x} + -\mathbf{x} = \mathbf{0}$.
5. For all vectors $\mathbf{x}$ and $\mathbf{y} \in V$ and all scalars $c \in \mathbf{R}$, $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$.
6. For all vectors $\mathbf{x} \in V$, and all scalars c and $d \in \mathbf{R}$, $(c + d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$.
7. For all vectors $\mathbf{x} \in V$, and all scalars c and $d \in \mathbf{R}$, $(cd)\mathbf{x} = c(d\mathbf{x})$.
8. For all vectors $\mathbf{x} \in V$, $1\mathbf{x} = \mathbf{x}$.

(1.1.6) Proposition. Let V be a vector space. Then

a) The zero vector $\mathbf{0}$ is unique.

b) For all $\mathbf{x} \in V$, $0\mathbf{x} = \mathbf{0}$.

c) For each $\mathbf{x} \in \mathbf{V}$, the additive inverse $-x$ is unique.

d) For all $\mathbf{x} \in V$, and all $c \in \mathbf{R}$, $(-c)\mathbf{x} = -(c\mathbf{x})$.

Basic Examples of Vector Spaces. For each example, identify:

(a) an expression for a general vector in the vector space
(b) the definitions of the addition and scalar multiplication in the vector space
(c) the zero vector (or the additive identity) in the vector space
(d) the additive inverse of a general vector in the vector space

1. $\mathbb{R}^n$

2. $P_n(\mathbb{R})$

3. $M_{m\times n}(\mathbb{R})$

Subspaces: Suppose V is a vector space. Explain what it means to say that a subset U of V is a subspace.

Subspace Theorem: Write down the theorem that you use to prove that a subset W of a vector space V is a subspace.

Ex. 1. (a) Let $V = \mathbb{R}^2$ and $W = \{(x, y) \in \mathbb{R}^2 \mid 6x + 5y = 3\}$. Determine whether or not W is a subspace of V and justify your answer.

(b) Give two more examples of subsets of $\mathbb{R}^2$: one that is a subspace and one that is not a subspace. Justify your answers.

Ex. 2. $V = P_2(\mathbb{R})$ be the vector space of polynomials of degree two or less. Determine whether or not each of the following sets W is a subspace of V. Justify your answers.
(a) $W = \{p \in P_2(\mathbb{R}) \mid p(0) + p'(0) = 0\}$
(b) $W = \{p \in P_2(\mathbb{R}) \mid p'(0) = 2\}$

Ex. 3. Let U and V be subspaces of a vector space W. Prove that

$$2U + 5V = \{2\vec{u} + 5\vec{v} \ : \ \vec{u} \in U, \vec{v} \in V\}$$

is a subspace of W.

Linear Combinations: Let V be a vector space and $S = \{v_1, \dots, v_n\}$ be a subset of V. Write down a general linear combination of the elements of S.

Span: The span of a nonempty set S is the set of all linear combination of elements of S. Write down the definition of Span(S) in set-builder notation. (Note: Span($\emptyset$) is defined to be $\{\vec{0}\}$.)

Ex. 4. Let W be a subspace of a vector space V and let S be a subset of W. Prove that Span(S) $\subseteq W$. (Note: Here you are being asked to prove a common theorem, which you could usually use without proof.)

Ex. 5. Let $V = P_2(\mathbb{R})$ be the vector space of polynomials with real coefficients of degree at most two and let $S = \{1, 1+x, 1+x+x^2\}$. Prove that Span(S) $= V$.

Linear Independence and Dependence: Let $S = \{v_1, \ldots, v_n\}$ be a set of vectors in a vector space V.

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Ex. 6. Is the following set of polynomials linearly independent? Explain your answer.

$$\{x^4, x^4 + x^3, x^4 + x^3 + x^2\}$$

Ex. 7. Let W_1 and W_2 be subspaces of a vector space V and assume that $W_1 \bigcap W_2 = \{\vec{0}\}$. Let $w_1 \in W_1$ and $w_2 \in W_2$ be such that $w_1 \neq \vec{0}$ and $w_2 \neq \vec{0}$. Prove that $\{w_1, w_2\}$ is linearly independent.

Ex. 8. Suppose that u and v are vectors in a vector space V. Prove that $\{u, v\}$ is linearly independent if and only if $\{u + v, u - v\}$ is linearly independent.

Basis: A subset S of a vector space V is a basis of V if

(1)

(2)

Dimension: The dimension of a vector space V is the number of elements in a basis for V. (It is a theorem that any two bases of V have the same number of elements.) If V has no finite basis, we say $\dim(V) = \infty$.

Ex. 9. Let $P_3(\mathbb{R})$ be the vector space of polynomials with real coefficients and degree at most three. Let $W = \{p \in P_3(\mathbb{R}) \mid p(0) = p''(0) \text{ and } p'(1) = 0\}$.

(a) Prove that W is a subspace of $P_3(\mathbb{R})$.

(b) Find a basis for W.

(c) What is the dimension of W?

Ex. 10.

(a) Give a basis for the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ 0 \\ -4 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -1 \\ 2 \\ -1 \end{bmatrix}.$$

(b) Give an example of a vector in $\mathbb{R}^4$ that is not in the subspace in part (a). Justify your answer.

Additional Problems

Ex. 11. Determine whether or not each of the following sets W is a subspace of the vector space V. Justify your answers.

(a) $V = P(\mathbb{R})$ be the vector space of all polynomials and $W = \{p \in P(\mathbb{R}) \mid p(2) = 0\}$.
(b) $V = P(\mathbb{R})$ be the vector space of all polynomials and $W = \{p \in P(\mathbb{R}) \mid p(2) = 1\}$.
(c) $V = \mathbb{R}^3$ and $W = \{(x, y, z) \in \mathbb{R}^3 \mid x^4 - y^2 = 0\}$.
(d) $V = M_{2\times 2}(\mathbb{R})$ be the vector space of 2×2 matrices with real coefficients and
$W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{R}) \mid 3a - 2d = 0 \right\}$.

Ex. 12. Let W_1 and W_2 be subspaces of a vector space V.

(a) Prove that $W_1 \bigcap W_2$ is a subspace of V.
(b) Give an example to show that $W_1 \bigcup W_2$ need not be a subspace of V.
(c) Is $W_1 \setminus W_2$ a subpace of V?

Ex. 13. Let S be a subset of a vector space V. Prove that $\text{Span}(S)$ is a subspace of V. (Note: This is another common theorem that you could usually use without proof.)

Ex. 14. Suppose that V is a vector space and u and v are vectors in V. Show that $\text{Span}(\{u, v\}) = \text{Span}(\{u + v, u - v\})$.

Ex. 15. Suppose that u, v, and w are vectors in a vector space V and that $\{u, v, w\}$ is linearly independent. Prove that $\{u + 2v, v + 2w, u + 2w\}$ is linearly independent.

Ex. 16. Let $W = \{(x, y, z) \in \mathbb{R}^3 : x + 2y + 3z = 0\}$.

(a) Prove that W is a subspace of $\mathbb{R}^3$.
(b) Find a basis of W.

Ex. 17. Let U and V be subspaces of a vector space W.

(a) Prove that $U + V = \{u + v : u \in U, v \in V\}$ is a subspace of W.
(b) Suppose $\{u_1, \dots u_m\}$ is a basis for U and $\{v_1, \dots v_n\}$ is a basis for V. Prove that $\{u_1, \dots, u_m, v_1, \dots v_n\}$ spans $U + V$.
(c) Prove that $\dim(U + V) \leq \dim(U) + \dim(V)$.
(d) Give an example of a vector space W with subspaces U and V where $\dim(U + V) < \dim(U) + \dim(V)$.