1. Elementary Vector Analysis

(from Stewart, *Calculus*, Chapters 12 and 13)

Vectors, Points, and Position Vectors: The position vector of a point P is the vector from the origin to P. For example, in $\mathbb{R}^3$, a vector $\vec{v} = \langle x, y, z\rangle$ is the position vector of the point $P = (x, y, z)$. In this way, we often identify points as vectors and vice versa.

Parametrized Curves

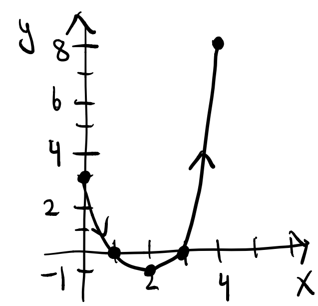
Parametrized Curves: Given a continuous vector function $\vec{r}(t) = \langle x(t), y(t)\rangle$ or $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, the set of points with position vector $\vec{r}(t)$ for some t defines a curve C. The components $x(t), y(t)$, and $z(t)$ are called parametric equations for the parametrized curve C. The variable t is called the parameter.

The tangent vector to a curve at the point $P = \vec{r}(t)$ is $\vec{r'}(t) = \langle x'(t), y'(t)\rangle$ or $\vec{r'}(t) = \langle x'(t), y'(t), z'(t)\rangle$. The line through the point $P = \vec{r}(t)$ parallel to the vector $\vec{r'}(t)$ is called the tangent line to C at P.

Ex. 1. (a) Sketch the curve C with parametric equations $x(t) = t + 1$, $y(t) = t^2 - 2t$, $-1 \le t \le 3$.
(b) Show that C is an arc of the parabola $y = x^2 - 4x + 3$ and use this fact to give another parametrization of C.

t	x	y
-1	0	3
0	1	0
1		
2		
3		

a)



b) We have $x = t+1$ and $y = t^2 - 2t$ so $t = x - 1$. Substituting into y,

$$y = (x-1)^2 - 2(x-1)$$
$$= x^2 - 2x + 1 - 2x + 2$$
$$y = x^2 - 4x + 3.$$

So C is the arc of this parabola for $0 \le x \le 4$.

Another parametrization of C is

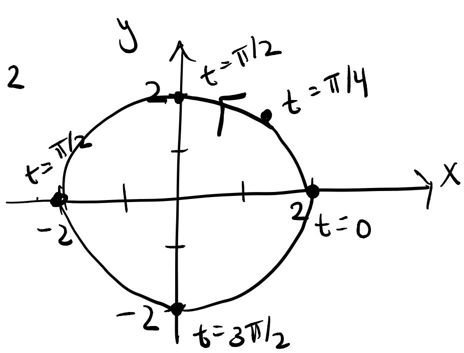
$x = t,\ y = t^2 - 4t + 3,\ 0 \le t \le 4.$

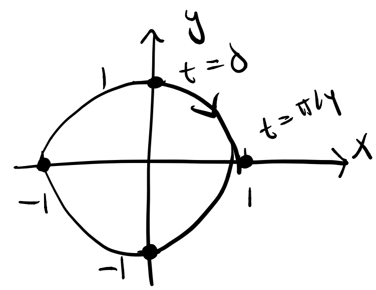
Ex. 2. What curve is represented by the parametric equations?

(a) $x(t) = 2\cos(t)$, $y(t) = 2\sin(t)$, $0 \le t \le 2\pi$

(b) $x(t) = \sin(2t)$, $y(t) = \cos(2t)$, $0 \le t \le 2\pi$

circle of radius 2 centered at the origin, traversed once counterclockwise.





unit circle centered at the origin, traversed twice clockwise.

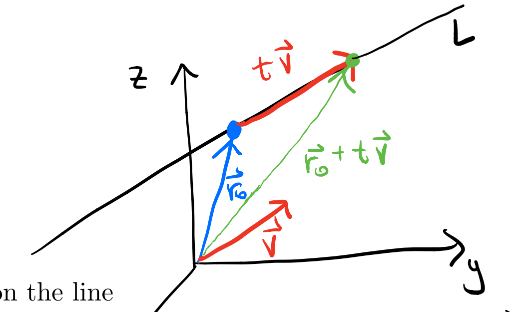
(c) Compute the tangent vector to the curve $\vec{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle$ and show that it is perpendicular to the curve at any point $(x(t), y(t))$.

The tangent vector is $\vec{r}\,'(t) = \langle -2\sin(t), 2\cos(t)\rangle$. For any t,

$\vec{r}(t) \cdot \vec{r}\,'(t) = \langle 2\cos t, 2\sin t\rangle \cdot \langle -2\sin t, 2\cos t\rangle = -4\sin t\cos t + 4\sin t\cos t = 0.$

So, $\vec{r}(t) \perp \vec{r}\,'(t)$ at any t.

Lines in $\mathbb{R}^2$ and $\mathbb{R}^3$



$\vec{r}(t) = \langle x_0, y_0, z_0\rangle + t\langle a,b,c\rangle$
$= \langle x_0+at, y_0+bt, z_0+ct\rangle$

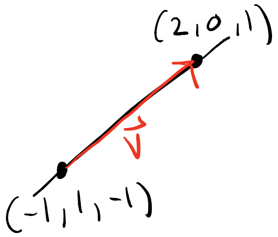
Vector Equation: $\vec{r}(t) = \vec{r_0} + t\vec{v}$

Parametric Equations: $x(t) = x_0 + at,\ y(t) = y_0 + bt,\ z(t) = z_0 + ct$

$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ are the position vectors of the points $(x(t), y(t), z(t))$ on the line
$\vec{r_0} = \langle x_0, y_0, z_0\rangle$ is the position vector of a fixed point $P_0 = (x_0, y_0, z_0)$ on the line
$\vec{v} = \langle a, b, c\rangle$ is a direction vector of the line, i.e. $\vec{v}$ is any vector parallel to the line

Rmk. For lines in $\mathbb{R}^2$, just delete all of the third coordinates above.

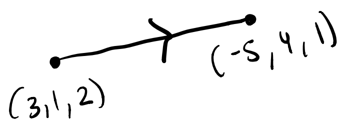
Ex. 3. Find parametric equations for the line that passes through the points $(2, 0, 1)$ and $(-1, 1, -1)$.



$\vec{v} = \langle 2-(-1),\ 0-1,\ 1-(-1)\rangle$ (terminal pt − initial pt)
$= \langle 3, -1, 2\rangle$
$\vec{r_0} = (-1, 1, -1)$

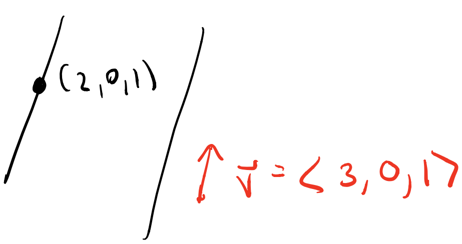
$x = -1+3t,\ y = 1-t,\ z = -1+2t$ (is one possible answer)

Ex. 4. Find parametric equations for the line segment from $(3, 1, 2)$ to $(-5, 4, 1)$.
start end



$\vec{v} = \langle -5-3,\ 4-1,\ 1-2\rangle$
$= \langle -8, 3, -1\rangle$
$\vec{r_0} = \langle 3, 1, 2\rangle$
$x = 3-8t,\ y = 1+3t,\ z = 2-t,\ 0 \le t \le 1$

Ex. 5. Find parametric equations for the line that passes through the point $(2, 0, 1)$ and is parallel to the line given by parametric equations $x(t) = 3t$, $y(t) = -5$, and $z(t) = t + 4$.



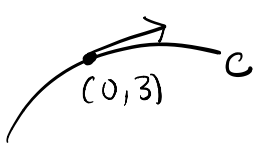
$\vec{r_0} = \langle 2, 0, 1\rangle$

$x = 2+3t,\ y = 0,\ z = 1+t$

Ex. 6. Find parametric equations for the tangent line to the parametrized curve $x(t) = t + 1$, $y(t) = t^2 - 2t$, at the point $(0, 3)$.

$\vec{v}$ = tangent vector at $(0,3)$

$\vec{r}(t) = \langle t+1, t^2-2t\rangle$
$\vec{r}\,'(t) = \langle 1, 2t-2\rangle$ is the tangent vector



At $(0,3)$, $x = 0 = t+1 \Rightarrow t = -1$ (Check $y = 1+2 = 3$ ✓)

So $\vec{v} = \vec{r}\,'(-1) = \langle 1, -4\rangle$

And $\vec{r_0} = \langle 0, 3\rangle$ so the tangent line is parametrized by

$x = t,\ y = 3-4t.$

Planes in $\mathbb{R}^3$

$\vec{n}\cdot(\vec{r}-\vec{r_0}) = \langle a,b,c\rangle \cdot \langle x-x_0, y-y_0, z-z_0\rangle$
$= a(x-x_0)+b(y-y_0)+c(z-z_0)$
$= 0$ bc. $n \perp \vec{r}-\vec{r_0}$

Vector Equation: $\vec{n}\cdot(\vec{r}-\vec{r_0}) = 0$

Scalar Equations: $\vec{r}-\vec{r_0}$ is always parallel to the plane

So $\vec{n} \perp \vec{r}-\vec{r_0}$

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

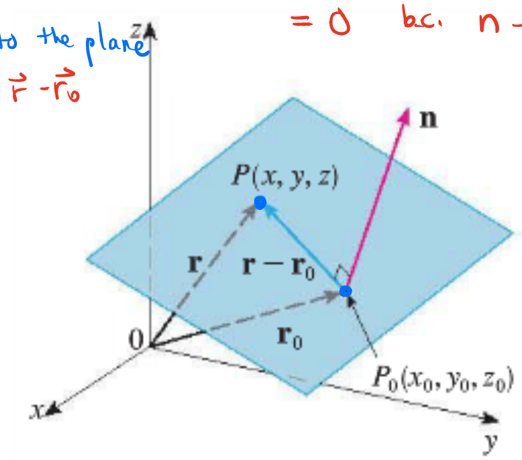
or

$$ax+by+cz=d$$

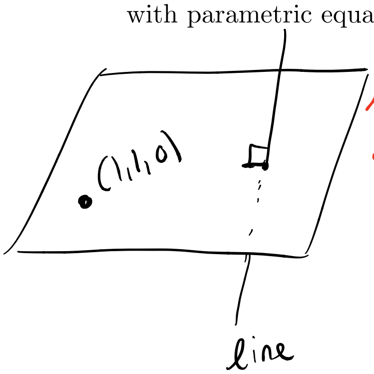
$\vec{r} = \langle x,y,z\rangle$ are the position vectors of points (x,y,z) in the plane

$\vec{r_0} = \langle x_0,y_0,z_0\rangle$ is the position vector of a fixed point $P_0 = (x_0,y_0,z_0)$ in the plane

$\vec{n} = \langle a,b,c\rangle$ is the normal vector, i.e. any vector orthogonal to the plane



Ex. 7. Find an equation of the plane that passes through the point $(1,1,0)$ and is perpendicular to the line with parametric equations $x(t) = 2+3t$, $y(t) = 1-t$, $z(t) = 2+2t$.



$\vec{v} = \langle 3,-1,2\rangle$

$\vec{v}$ is parallel to the line and hence also perpendicular to the plane.

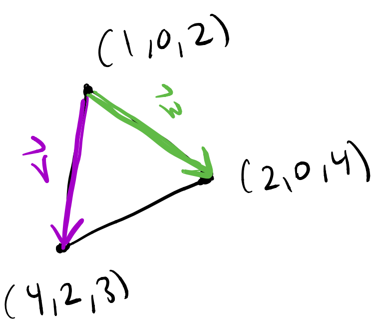
So let $\vec{n} = \vec{v} = \langle 3,-1,2\rangle$.

and $\vec{r_0} = \langle 1,1,0\rangle$. Then

$3(x-1) + -1(y-1) + 2(z) = 0$ is an eq. of the plane.

or $3x - y + 2z = 2$

Ex. 8. Find an equation of the plane that passes through the points $(1,0,2)$, $(4,2,3)$, and $(2,0,4)$.



The three points determine a triangle and hence a plane containing that triangle.

The vectors $\vec{v} = \langle 4-1, 2-0, 3-2\rangle = \langle 3,2,1\rangle$ and

$\vec{w} = \langle 2-1, 0-0, 4-2\rangle = \langle 1,0,2\rangle$

are parallel to the plane and so $\vec{n} = \vec{v}\times\vec{w}$ is $\perp$ to plane.

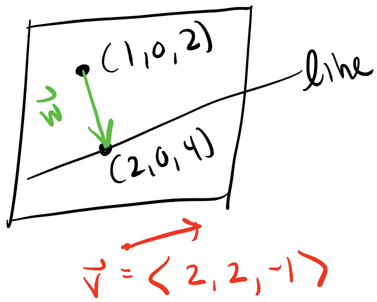
$$\vec{v}\times\vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & 1 \\ 1 & 0 & 2 \end{vmatrix} = (2\cdot2-1\cdot0)\vec{i} - (3\cdot2-1\cdot1)\vec{j} + (3\cdot0-1\cdot2)\vec{k}$$
$$= 4\vec{i} - 5\vec{j} + {}^-2\vec{k}$$
$$= \langle 4, -5, {}^-2\rangle$$

Then an eq. of the plane is $4(x-1) - 5(y) - 2(z-2) = 0$.

or $4x - 5y - 2z = 0$

Ex. 9. Find an equation of the plane that passes through the point $(1, 0, 2)$ and contains the line

$$x = 2t + 2, y = 2t, z = 4 - t.$$



The vector $\vec{v} = \langle 2,2,-1 \rangle$ is parallel to the plane.

Letting $t=0$, the point $(2,0,4)$ is on the line and hence in the plane. Then the vector $\vec{w} = \langle 2-1, 0-0, 4-2 \rangle = \langle 1,0,2 \rangle$ is also parallel to the plane. So, $\vec{n} = \vec{v} \times \vec{w}$ is $\perp$ to the plane.

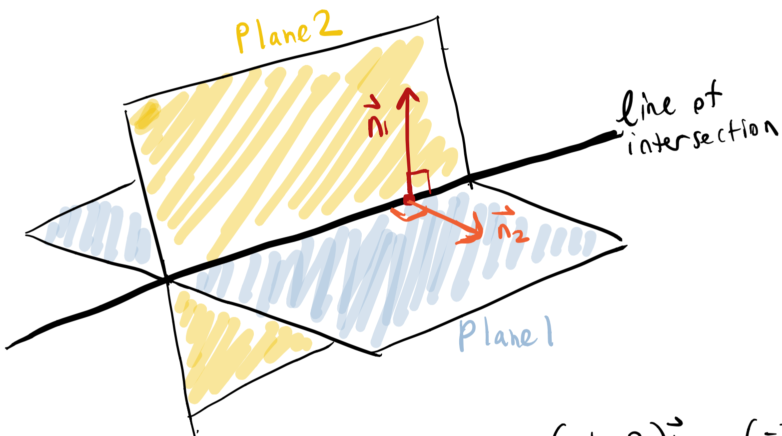
$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & -1 \\ 1 & 0 & 2 \end{vmatrix} = 4\vec{i} - (4+1)\vec{j} + -2\vec{k}$$

$$= \langle 4, -5, -2 \rangle$$

Then an eq. of the plane is $4(x-1) + -5y - 2(z-2) = 0$.

or $4x - 5y - 2z = 0$

Ex. 10. Find parametric equations for the line of intersection for the two planes $2x + y + z = 8$ and $x + 2y - z = 1$.



$\vec{n_1} = \langle 2,1,1 \rangle$ is $\perp$ to plane 1 and hence also the the line of intersection, since it is contained in plane 1.

Similarly, $\vec{n_2} = \langle 1,2,-1 \rangle$ is also perpendicular to the line. Hence, $\vec{v} = \vec{n_1} \times \vec{n_2}$ is parallel to the line.

$$\vec{n_1} \times \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = (-1-2)\vec{i} - (-2-1)\vec{j} + (4-1)\vec{k}$$

$$= \langle -3, 3, 3 \rangle$$

Now a point on the line should be on both planes. Let's try taking $x=0$ to find the int. of the line with the yz-plane. We have $y+z=8$ and $2y-z=1$

$\Rightarrow 3y = 9 \Rightarrow y=3 \Rightarrow z=5$. Then $(0,3,5)$ is a point on the line.

So parametric eqs are $x = -3t$, $y = 3t+3$, $z = 3t+5$.