1. Vector Spaces and Subspaces

(References: Comps Study Guide for Linear Algebra Section 1; Damiano & Little, *A Course in Linear Algebra*, Chapter 1)

Vector Spaces

(1.1.1) Definition. A (real) *vector space* is a set V (whose elements are called *vectors* by analogy with the first example we considered) together with

a) an operation called *vector addition*, which for each pair of vectors $\mathbf{x}, \mathbf{y} \in V$ produces another vector in V denoted $\mathbf{x} + \mathbf{y}$, and

b) an operation called *multiplication by a scalar* (a real number), which for each vector $\mathbf{x} \in V$, and each scalar $c \in \mathbf{R}$ produces another vector in V denoted $c\mathbf{x}$.

Furthermore, the two operations must satisfy the following *axioms:*

properties of the operations

1. For all vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z} \in V$, $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.
2. For all vectors $\mathbf{x}$ and $\mathbf{y} \in V$, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.
3. There exists a vector $\mathbf{0} \in V$ with the property that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for all vectors $\mathbf{x} \in V$. existence of additive identity
4. For each vector $\mathbf{x} \in V$, there exists a vector denoted $-\mathbf{x}$ with the property that $\mathbf{x} + -\mathbf{x} = \mathbf{0}$. existence of additive inverses
5. For all vectors $\mathbf{x}$ and $\mathbf{y} \in V$ and all scalars $c \in \mathbf{R}$, $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$.
6. For all vectors $\mathbf{x} \in V$, and all scalars c and $d \in \mathbf{R}$, $(c + d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$.
7. For all vectors $\mathbf{x} \in V$, and all scalars c and $d \in \mathbf{R}$, $(cd)\mathbf{x} = c(d\mathbf{x})$.
8. For all vectors $\mathbf{x} \in V$, $1\mathbf{x} = \mathbf{x}$.

(1.1.6) Proposition. Let V be a vector space. Then

a) The zero vector $\mathbf{0}$ is unique.

b) For all $\mathbf{x} \in V$, $0\mathbf{x} = \mathbf{0}$. scalar (i.e. the real # zero) vector (the zero vector in V)

c) For each $\mathbf{x} \in \mathbf{V}$, the additive inverse $-x$ is unique.

d) For all $\mathbf{x} \in V$, and all $c \in \mathbf{R}$, $(-c)\mathbf{x} = -(c\mathbf{x})$.

Basic Examples of Vector Spaces. For each example, identify:

(a) an expression for a general vector in the vector space
(b) the definitions of the addition and scalar multiplication in the vector space
(c) the zero vector (or the additive identity) in the vector space
(d) the additive inverse of a general vector in the vector space

1. $\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R}\}$

2. $P_n(\mathbb{R}) = \{a_0 + a_1 x + \ldots + a_n x^n \mid a_i \in \mathbb{R}\}$

3. $M_{m\times n}(\mathbb{R})$

$$= \left\{ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & \ddots & \\ a_{m1} & & & a_{mn} \end{pmatrix} \middle| \ a_{ij} \in \mathbb{R} \right\}$$

Subspaces: Suppose V is a vector space. Explain what it means to say that a subset U of V is a subspace.

This means that U is also a vector space, with the same addition and scalar multiplication as V.

(Note: this means that U must also contain the zero vector of V.)

Subspace Theorem: Write down the theorem that you use to prove that a subset W of a vector space V is a subspace.

A subset W of a vector space V is a subspace if:

1. $W \neq \emptyset$ (non-empty, usually checked by showing $\vec{0} \in W$)

2. For all $x, y \in W$ and $c \in \mathbb{R}$, $cx + y \in W$. (closure under addition & scalar multi.)

or equivalently 2a. $\forall x, y \in W$, $x + y \in W$

2b. $\forall x \in W$ and $\forall c \in \mathbb{R}$, $cx \in W$

Ex. 1. (a) Let $V = \mathbb{R}^2$ and $W = \{(x, y) \in \mathbb{R}^2 \mid 6x + 5y = 3\}$. Determine whether or not W is a subspace of V and justify your answer.

The zero vector in V is $\vec{0} = (0,0)$, but $6(0) + 5(0) = 0 \neq 3$ so $\vec{0} \notin W$. Hence, W is not a subspace of V.

(b) Give two more examples of subsets of $\mathbb{R}^2$: one that is a subspace and one that is not a subspace. Justify your answers.

1. The set $W_1 = \{(x,y) \in \mathbb{R}^2 \mid 6x + 5y = 1\}$ is not a subspace of $\mathbb{R}^2$ because it does not contain the zero vector $\vec{0} = (0,0)$.

2. The set $W_2 = \{(x,y) \in \mathbb{R}^2 \mid 6x + 5y = 0\}$ is a subspace of $\mathbb{R}^2$. To see this, first note that $\vec{0} = (0,0) \in W_2$. Next, suppose that $c \in \mathbb{R}$ and $(x_1, y_1), (x_2, y_2) \in W_2$. Then $6x_1 + 5y_1 = 0$ and $6x_2 + 5y_2 = 0$, which implies that $6(cx_1 + x_2) + 5(cy_1 + y_2) = c(6x_1 + 5y_1) + (6x_2 + 5y_2) = 0$. Thus, $(cx_1 + x_2, cy_1 + y_2) = c(x_1, y_1) + (x_2, y_2) \in W_2$ and W_2 is a subspace of $\mathbb{R}^2$.

Ex. 2. $V = P_2(\mathbb{R})$ be the vector space of polynomials of degree two or less. Determine whether or not each of the following sets W is a subspace of V. Justify your answers.

(a) $W = \{p \in P_2(\mathbb{R}) \mid p(0) + p'(0) = 0\}$
(b) $W = \{p \in P_2(\mathbb{R}) \mid p'(0) = 2\}$

a) Let $z(x) = 0 \ \forall x$ be the zero vector in $P_2(\mathbb{R})$. Then $z(0) + z'(0) = 0 + 0 = 0$. So $z \in W$. Now let $c \in \mathbb{R}$ and $p, q \in W$. So $p(0) + p'(0) = 0$ and $q(0) + q'(0) = 0$. Then $(cp+q)(0) + (cp+q)'(0) = cp(0) + q(0) + cp'(0) + q'(0)$

$$= c(p(0) + p'(0)) + (q(0) + q'(0)) = 0 + 0 = 0.$$

So $cp + q \in W$. Thus, W is a subspace of V.

b) Let $z(x) = 0 \ \forall x$ be the zero vector in $P_2(\mathbb{R})$. Then $z'(0) = 0 \neq 2$ so $z \notin W$. Thus, W is not a subspace of V.

Ex. 3. Let U and V be subspaces of a vector space W. Prove that

$$2U + 5V = \{2\vec{u} + 5\vec{v} \ : \ \vec{u} \in U, \vec{v} \in V\}$$

is a subspace of W.

Since U and V are subspaces of W, $\vec{0} \in U$ and $\vec{0} \in V$. Then $2 \cdot \vec{0} + 5 \cdot \vec{0} = \vec{0} \in 2U + 5V$. Let $c \in \mathbb{R}$ and $2\vec{u}_1 + 5\vec{v}_1$, $2\vec{u}_2 + 5\vec{v}_2$ be in $2U + 5V$ for $\vec{u}_1, \vec{u}_2 \in U$ and $\vec{v}_1, \vec{v}_2 \in V$. Then $c(2\vec{u}_1 + 5\vec{v}_1) + (2\vec{u}_2 + 5\vec{v}_2) = 2(c\vec{u}_1 + \vec{u}_2) + 5(c\vec{v}_1 + \vec{v}_2)$. Now since $\vec{u}_1, \vec{u}_2 \in U$ and U is a subspace, $c\vec{u}_1 + \vec{u}_2 \in U$ as well. Similarly, since V is a subspace, $c\vec{v}_1 + \vec{v}_2 \in V$. Then $2(c\vec{u}_1 + \vec{u}_2) + 5(c\vec{v}_1 + \vec{v}_2) \in 2U + 5V$ and hence $c(2\vec{u}_1 + 5\vec{v}_1) + (2\vec{u}_2 + 5\vec{v}_2) \in 2U + 5V$. Thus, $2U + 5V$ is a subspace of W.

Linear Combinations: Let V be a vector space and $S = \{v_1, \ldots, v_n\}$ be a subset of V. Write down a general linear combination of the elements of S.

$$a_1 v_1 + \cdots + a_n v_n, \quad a_i \in \mathbb{R}$$

Span: The span of a nonempty set S is the set of all linear combination of elements of S. Write down the definition of Span(S) in set-builder notation. (Note: Span($\emptyset$) is defined to be $\{\vec{0}\}$.)

$$\mathrm{Span}(S) = \{ a_1 v_1 + \cdots + a_n v_n \mid a_1, \ldots, a_n \in \mathbb{R} \}$$

To show $A \subseteq B$, let $x \in A$ and show that $x \in B$.

Ex. 4. Let W be a subspace of a vector space V and let S be a subset of W. Prove that Span(S) $\subseteq W$. (Note: Here you are being asked to prove a common theorem, which you could usually use without proof.)

Let $x \in \mathrm{Span}(S)$. Then $x = a_1 v_1 + \cdots + a_n v_n$ for some $a_1, \ldots, a_n \in \mathbb{R}$ and $v_1, \ldots, v_n \in S$. Since $S \subseteq W$, $v_1, \ldots, v_n \in W$ and since W is a subspace, $a_1 v_1 + \cdots + a_n v_2 \in W$ as well. Thus, $x \in W$. Therefore, $\mathrm{Span}(S) \subseteq W$.

Ex. 5. Let $V = P_2(\mathbb{R})$ be the vector space of polynomials with real coefficients of degree at most two and let $S = \{1, 1+x, 1+x+x^2\}$. Prove that Span(S) $= V$. To show $A = B$, show $A \subseteq B$ and $B \subseteq A$.

$\subseteq$ Since V is a vector space and $S \subseteq V$, we know that $\mathrm{Span}(S) \subseteq V$.

$\supseteq$ Suppose that $p \in V = P_2(\mathbb{R})$. So $p(x) = a_0 + a_1 x + a_2 x^2$ for some $a_0, a_1, a_2 \in \mathbb{R}$.

Scratchwork We want to show that $p \in \mathrm{Span}(S)$, i.e. $p(x) = c_1(1) + c_2(1+x) + c_3(1+x+x^2)$ for some $c_1, c_2, c_3 \in \mathbb{R}$. To find the c_i, we set up a system of equations.

$$a_0 + a_1 x + a_2 x^2 = c_1 + c_2(1+x) + c_3(1+x+x^2)$$
$$= (c_1 + c_2 + c_3) + (c_2 + c_3)x + c_3 x^2$$

$$\Longrightarrow \begin{aligned} c_1 + c_2 + c_3 &= a_0 \\ c_2 + c_3 &= a_1 \\ c_3 &= a_2 \end{aligned} \qquad \begin{aligned} c_2 &= a_1 - a_2 \\ c_1 &= a_0 - (a_1 - a_2) - a_2 = a_0 - a_1 \end{aligned}$$

Then $(a_0 - a_1)(1) + (a_1 - a_2)(1+x) + a_2(1+x+x^2) = a_0 + a_1 x + a_2 x^2 = p(x)$.

So, $p \in \mathrm{Span}(S)$ and $V \subseteq \mathrm{Span}(S)$. Thus, $\mathrm{Span}(S) = V$.

Linear Independence and Dependence: Let $S = \{v_1, \ldots, v_n\}$ be a set of vectors in a vector space V.

The set S is *linearly dependent* if

there exist $a_1, \ldots, a_n \in \mathbb{R}$ not all zero such that $a_1 v_1 + \ldots + a_n v_n = \vec{0}$.

The set S is *linearly independent* if

for any $a_1, \ldots, a_n \in \mathbb{R}$ satisfying $a_1 v_1 + \ldots + a_n v_n = 0$, we have $a_1 = \ldots = a_n = 0$.

Ex. 6. Is the following set of polynomials linearly independent? Explain your answer.

$$\{x^4, x^4 + x^3, x^4 + x^3 + x^2\}$$

Let $a_1 x^4 + a_2(x^4 + x^3) + a_3(x^4 + x^3 + x^2) = 0$ for some $a_1, a_2, a_3 \in \mathbb{R}$. Then

$(a_1 + a_2 + a_3)x^4 + (a_2 + a_3)x^3 + a_3 x^2 = 0$ which implies that

$$a_1 + a_2 + a_3 = 0$$
$$a_2 + a_3 = 0$$
$$a_3 = 0.$$

Substituting $a_3 = 0$ gives $a_2 = 0$ and $a_1 = 0$. Thus, $a_1 = a_2 = a_3 = 0$.

Therefore, $\{x^4, x^4 + x^3, x^4 + x^3 + x^2\}$ is linearly independent.

Ex. 7. Let W_1 and W_2 be subspaces of a vector space V and assume that $W_1 \bigcap W_2 = \{\vec{0}\}$. Let $w_1 \in W_1$ and $w_2 \in W_2$ be such that $w_1 \neq \vec{0}$ and $w_2 \neq \vec{0}$. Prove that $\{w_1, w_2\}$ is linearly independent.

Let $a_1, a_2 \in \mathbb{R}$ be such that $a_1 w_1 + a_2 w_2 = \vec{0}$. Then $a_1 w_1 = -a_2 w_2$.

Now $w_1 \in W_1$ and since W_1 is a subspace, $a_1 w_1 \in W_1$ as well,

Similarly, $-a_2 w_2 \in W_2$. Hence, $a_1 w_1 = -a_2 w_2 \in W_1$ and

$a_1 w_1 = -a_2 w_2 \in W_2$, ie $a_1 w_1 = -a_2 w_2 \in W_1 \cap W_2$. Since $W_1 \cap W_2 = \{\vec{0}\}$,

we have $a_1 w_1 = \vec{0}$ and $-a_2 w_2 = \vec{0}$. But since neither w_1 nor w_2

is $\vec{0}$, it must be that $a_1 = 0$ and $-a_2 = 0$. Then $a_1 = a_2 = 0$.

Thus, $\{w_1, w_2\}$ is linearly independent.

Ex. 8. Suppose that u and v are vectors in a vector space V. Prove that $\{u, v\}$ is linearly independent if and only if $\{u+v, u-v\}$ is linearly independent.

$\Rightarrow$ First suppose that $\{u, v\}$ is linearly independent.

Note: we want to show that $\{u+v, u-v\}$ is lin ind, which means that we need to show for any $a_1, a_2 \in \mathbb{R}$ satisfying $a_1(u+v) + a_2(u-v) = \vec{0}$, we have $a_1 = a_2 = 0$.

Let $a_1, a_2 \in \mathbb{R}$ such that $a_1(u+v) + a_2(u-v) = \vec{0}$. Then $(a_1+a_2)\vec{u} + (a_1-a_2)\vec{v} = \vec{0}$ and since $\{u, v\}$ is linearly independent, we have $a_1 + a_2 = 0$ and $a_1 - a_2 = 0$. Solving this system by adding the left and right hand sides gives $2a_1 = 0$. Hence, $a_1 = 0$ and substituting gives $a_2 = 0$ as well. Thus, $a_1 = a_2 = 0$ and so $\{u+v, u-v\}$ must be linearly independent.

$\Leftarrow$ Next suppose that $\{u+v, u-v\}$ is linearly independent. To show that $\{u, v\}$ is also linearly independent, let $a_1, a_2 \in \mathbb{R}$ be such that $a_1 u + a_2 v = 0$.

Scratchwork: We want to transform this to an equation of the form $c_1(u+v) + c_2(u-v) = 0$ so we can use the linear independence of $\{u+v, u-v\}$. To do this, we set

$$a_1 u + a_2 v = c_1(u+v) + c_2(u-v) = (c_1+c_2)u + (c_1-c_2)v$$

$$\Rightarrow \begin{array}{l} c_1 + c_2 = a_1 \\ c_1 - c_2 = a_2 \end{array} \Rightarrow \begin{array}{l} c_1 = \frac{1}{2}(a_1+a_2) \\ c_2 = \frac{1}{2}(a_1-a_2) \end{array}$$

Then $\frac{1}{2}(a_1+a_2)(u+v) + \frac{1}{2}(a_1-a_2)(u-v) = a_1 u + a_2 v = 0$. Since $\{u+v, u-v\}$ is linearly independent, we must have $\frac{1}{2}(a_1+a_2) = 0$ and $\frac{1}{2}(a_1-a_2) = 0$. But then $a_1 + a_2 = 0$ and $a_1 - a_2 = 0$ which we have shown above implies that $a_1 = a_2 = 0$. So, $\{u, v\}$ is linearly independent as well.

Basis: A subset S of a vector space V is a basis of V if

(1) S is linearly independent and

(2) $\mathrm{Span}(S) = V$.

Dimension: The dimension of a vector space V is the number of elements in a basis for V. (It is a theorem that any two bases of V have the same number of elements.) If V has no finite basis, we say $\dim(V) = \infty$.

Ex. 9. Let $P_3(\mathbb{R})$ be the vector space of polynomials with real coefficients and degree at most three. Let $W = \{p \in P_3(\mathbb{R}) \mid p(0) = p''(0) \text{ and } p'(1) = 0\}$.

(a) Prove that W is a subspace of $P_3(\mathbb{R})$.

Let $z \in P_3(\mathbb{R})$ be the constant polynomial, $z(x) = 0$ for all x. Then $z(0) = z''(0) = 0$ and $z'(1) = 0$. Hence, $z \in W$. Next let $c \in \mathbb{R}$ and $p, q \in W$. Then $p(0) = p''(0)$, $p'(1) = 0$, $q(0) = q''(0)$, and $q'(1) = 0$. Then $(cp+q)(0) = cp(0) + q(0) = cp''(0) + q''(0) = (cp'' + q'')(0)$ and $(cp+q)'(1) = cp'(1) + q'(1) = c \cdot 0 + 0 = 0$. So $cp+q \in W$ as well. Thus, W is a subspace of $P_3(\mathbb{R})$.

(b) Find a basis for W.

Let $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \in W$ for $a_0, a_1, a_2, a_3 \in \mathbb{R}$. Then $p'(x) = a_1 + 2a_2 x + 3a_3 x^2$ and $p''(x) = 2a_2 + 6a_3 x$. So $p(0) = a_0$ and $p''(0) = 2a_2$. Then the condition $p(0) = p''(0)$ implies that $a_0 = 2a_2$. And from $p'(1) = 0$ we have $a_1 + 2a_2 + 3a_3 = 0$. Hence, we have $a_0 = 2a_2$ and $a_1 = -2a_2 - 3a_3$ with a_2 and a_3 free. Substituting, $p(x)$ must be of the form $p(x) = 2a_2 + (-2a_2 - 3a_3)x + a_2 x^2 + a_3 x^3$ for some $a_2, a_3 \in \mathbb{R}$. Rearranging, we have $p(x) = a_2(2 - 2x + x^2) + a_3(-3x + x^3)$. Hence a basis for W is $\{2 - 2x + x^2, -3x + x^3\}$.

(c) What is the dimension of W?

By part b, $\dim(W) = 2$.

Ex. 10.

(a) Give a basis for the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\text{Let} \quad v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -2 \\ 0 \\ -4 \\ 2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 2 \\ -1 \\ 2 \\ -1 \end{bmatrix}.$$

(b) Give an example of a vector in $\mathbb{R}^4$ that is not in the subspace in part (a). Justify your answer.

a) By inspection, $v_2 = -2v_1$ so $v_2 \in \text{Span}\{v_1, v_3, v_4\}$ and can be immediately eliminated. For the remaining three vectors, we row-reduce the matrix

$$\begin{bmatrix} | & | & | \\ v_1 & v_3 & v_4 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{bmatrix} \xrightarrow[-2R1+R3 \to R3]{R1+R4 \to R4} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[-2R2+R3 \to R3]{R2+R4 \to R4} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The row-reduced matrix has pivots in the first and second columns and so the columns of the original matrix are spanned by v_1 and v_3. Thus, the subspace is $\text{Span}(\{v_1, v_3\})$. And clearly v_1 and v_3 are not scalar multiples of each other since the first entry in v_1 is 1 and the first entry in v_3 is 0. Hence $\{v_1, v_3\}$ is linearly independent. Thus, a basis of this subspace is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ -1 \end{pmatrix} \right\}$.

Note: other possible answers are $\{v_1, v_2\}$ and $\{v_2, v_3\}$.

b) The vector $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ is not in this subspace. To see this, suppose that

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 2 \\ -1 \end{pmatrix}$$

for some $c_1, c_2 \in \mathbb{R}$. Then we have the system of equations

$$\begin{aligned} c_1 &= 0 \\ c_2 &= 0 \\ 2c_1 + 2c_2 &= 0 \\ -c_1 - c_2 &= 1 \end{aligned}$$

which is inconsistent since $c_1 = c_2 = 0$ implies that $-c - c_2 = 0$, not 1. Hence, $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \notin \text{Span}(\{v_1, v_3\})$.

Additional Problems

Ex. 11. Determine whether or not each of the following sets W is a subspace of the vector space V. Justify your answers.

(a) $V = P(\mathbb{R})$ be the vector space of all polynomials and $W = \{p \in P(\mathbb{R}) \mid p(2) = 0\}$. Yes

(b) $V = P(\mathbb{R})$ be the vector space of all polynomials and $W = \{p \in P(\mathbb{R}) \mid p(2) = 1\}$. No, $\vec{0} \notin W$

(c) $V = \mathbb{R}^3$ and $W = \{(x, y, z) \in \mathbb{R}^3 \mid x^4 - y^2 = 0\}$. No, $(1,1,0) \in W$ but $(2,2,0) \notin W$

(d) $V = M_{2\times 2}(\mathbb{R})$ be the vector space of 2×2 matrices with real coefficients and

$$W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{R}) \mid 3a - 2d = 0 \right\}.$$ Yes

Ex. 12. Let W_1 and W_2 be subspaces of a vector space V.

see study guide

(a) Prove that $W_1 \bigcap W_2$ is a subspace of V.

(b) Give an example to show that $W_1 \bigcup W_2$ need not be a subspace of V.

(c) Is $W_1 \setminus W_2$ a subpace of V? No, $\vec{0} \notin W_1 \setminus W_2$

Ex. 13. Let S be a subset of a vector space V. Prove that $\text{Span}(S)$ is a subspace of V. (Note: This is another common theorem that you could usually use without proof.) see Thm 1.3.4 in text

Ex. 14. Suppose that V is a vector space and u and v are vectors in V. Show that $\text{Span}(\{u, v\}) = \text{Span}(\{u + v, u - v\})$. Use the scratchwork from Ex. 8

Ex. 15. Suppose that u, v, and w are vectors in a vector space V and that $\{u, v, w\}$ is linearly independent. Prove that $\{u + 2v, v + 2w, u + 2w\}$ is linearly independent. similar to Ex. 8

~~**Ex. 16.** Let V be a vector space. Suppose that $\{u, v, w\}$ is a basis of V. Is the set $\{u + 2v, v + 2w, u + 2w\}$ also a basis of V? Justify your answer. Yes, use Ex. 15~~ moved

Ex. 16. Let $W = \{(x, y, z) \in \mathbb{R}^3 : x + 2y + 3z = 0\}$.

(a) Prove that W is a subspace of $\mathbb{R}^3$. Similar to Ex. 1

(b) Find a basis of W. $x = -2y - 3z$ with y, z free $\Rightarrow (-2y-3z, y, z) = y(-2,1,0) + z(-3,0,1)$
$\Rightarrow$ Basis $= \{(-2,1,0), (-3,0,1)\}$

Ex. 17. Let U and V be subspaces of a vector space W.

(a) Prove that $U + V = \{u + v : u \in U, v \in V\}$ is a subspace of W. similar to Ex. 3

(b) Suppose $\{u_1, \ldots u_m\}$ is a basis for U and $\{v_1, \ldots v_n\}$ is a basis for V. Prove that $\{u_1, \ldots, u_m, v_1, \ldots v_n\} = S$ spans $U + V$. For $u+v \in U+V$, $u = c_1u_1 + \ldots + c_mu_m$ and $v = a_1v_1 + \ldots + a_nv_n$ for some $c_i, a_i \in \mathbb{R}$.

(c) Prove that $\dim(U + V) \le \dim(U) + \dim(V)$. You can reduce S to a basis of $U+V$.

(d) Give an example of a vector space W with subspaces U and V where $\dim(U + V) < \dim(U) + \dim(V)$.

$W = \mathbb{R}^2$, $U = \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} = V$