1. Addendum: More on Basis and Dimension

(This material appears in Section 2 of the Study Guide, but in Chapter 1 of the text.)

Theorems about dimension from Chapter 1

- If $X \subseteq V$ is a subspace, then $\dim(X) \leq \dim(V)$. Moreover, if $dim(V) < \infty$, then $\dim(X) = \dim(V)$ if and only if $X = V$.
- Let V be a vector space with $\dim(V) = n$, and let $S \subseteq V$ be a set of m distinct vectors in V.
  - If $m < n$, then S cannot span V.
  - If $m > n$, then S cannot be linearly independent.

Challenge: Explain how the following "two-out-of-three theorem" follows from the results above.

Two-out-of-three Theorem for Sets: Let V be a vector space, and let $S \subseteq V$ be a set of n distinct vectors in V. If any two of the following conditions hold, then all three hold (and S is a basis for V).

(1) S is linearly independent.
(2) S spans V.
(3) $\dim(V) = n$.

1 & 2 $\Rightarrow$ S is a basis of V $\Rightarrow$ $\dim(V) = n$

1 & 3 $\Rightarrow$ S is a basis for Span(S) $\Rightarrow$ $\dim(\text{Span}(S)) = n = \dim(V)$

Span(S) subspace of V $\Rightarrow$ $\text{Span}(S) = V$

2 & 3 : Suppose S is not linearly ind. Then $\exists\, a_1, \ldots, a_n \in \mathbb{R}$ not all zero s.t. $a_1 v_1 + \ldots + a_n v_n = \vec{0}$. Suppose $a_k \neq 0$. Then $v_k = -\frac{1}{a_k}(v_1 + \ldots + v_{k-1} + v_{k+1} + \ldots + v_n)$, i.e. $v_k \in \text{Span}(S')$ where $S' = S \setminus \{v_k\}$. It then follows that $\text{Span}(S') = \text{Span}(S) = V$. But then V has a spanning set of size $n-1 < n = \dim(V)$, which is a contradiction. So, S must be lin. ind.

Ex. 1. Let V be a vector space. Is the set $\{x, 1+x, -1-2x+x^2\}$ a basis of $P_2(\mathbb{R})$? Justify your answer.

Call this set S. First we check if S is linearly independent.

Let $a_1, a_2, a_3 \in \mathbb{R}$ be such that $a_1 x + a_2(1+x) + a_3(-1-2x+x^2) = 0$.

Combining like terms, $(a_2 - a_3) + (a_1 + a_2 - 2a_3)x + a_3 x^2 = 0$.

Then we must have
$$\begin{aligned} a_1 + a_2 - 2a_3 &= 0 \\ a_2 - a_3 &= 0 \\ a_3 &= 0 \end{aligned}$$
which implies that $a_2 = 0$ and $a_3 = 0$.

Thus, $a_1 = a_2 = a_3 = 0$ and so S is linearly independent. Since $\dim(P_2(\mathbb{R})) = 3$ and S has 3 elements, S must span $P_2(\mathbb{R})$ as well. Therefore, S is a basis for $P_2(\mathbb{R})$.

2. Linear Transformations

(References: Comps Study Guide for Linear Algebra Section 2;
Damiano & Little, *A Course in Linear Algebra*, Chapter 2)

Linear Transformations: Let V and W be vector spaces and $T : V \to W$. Write down the condition(s) that T must satisfy in order to be linear.

For all $x, y \in V$ and $c \in \mathbb{R}$, we must have $\underbrace{T(cx+y) = cT(x) + T(y)}$.

This single condition can be replaced with ① $T(x+y) = T(x) + T(y)$
and ② $T(cx) = cT(y)$.

Rmk: We say that T respects (or preserves) the addition & scalar multiplication.

Ex. 2. Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. Suppose that $T(1,2) = (3,4)$ and $T(2,3) = (1,1)$. Find $T(x,y)$ for $(x,y) \in \mathbb{R}^2$.

We know $T(1,2)$ and $T(2,3)$ so we write (x,y) in terms of these vectors.

Set $(x,y) = c_1(1,2) + c_2(2,3) = (c_1 + 2c_2, 2c_1 + 3c_2)$. Then

$$\begin{cases} c_1 + 2c_2 = x \\ 2c_1 + 3c_2 = y \end{cases} \xrightarrow{-2E1 + E2 \to E2} \begin{cases} c_1 + 2c_2 = x \\ -c_2 = -2x + y \end{cases} \Rightarrow \begin{aligned} c_2 &= 2x - y \\ c_1 &= x - 2(2x-y) \\ &= -3x + 2y \end{aligned}$$

So, $(x,y) = (-3x+2y)(1,2) + (2x-y)(2,3)$. Applying T, we have

$$\begin{aligned} T(x,y) &= T[(-3x+2y)(1,2) + (2x-y)(2,3)] \\ &= (-3x+2y)T(1,2) + (2x-y)T(2,3) \\ &= (-3x+2y)(3,4) + (2x-y)(1,1). \\ &= (-9x+6y, -12x+8y) + (2x-y, 2x-y) \\ &= (-7x+5y, -10x+7y). \end{aligned}$$

Thus, $T(x,y) = (-7x+5y, -10x+7y)$.

Ex. 3. Let $T : V \to W$ be a linear transformation between vector spaces V and W. Prove that $T(\vec{0}_V) = \vec{0}_W$. (Note: This is a very commonly used result, which you should remember.)

Since $0_V + 0_V = 0_V$, we have $T(0_V + 0_V) = T(0_V)$. Since T is linear, this becomes $T(0_V) + T(0_V) = T(0_V)$. Adding $-T(0_V)$ to both sides,

$T(0_V) + T(0_V) + -T(0_V) = T(0_V) + -T(0_V)$ or

$T(0_V) + 0_W = 0_W$. Since $T(0_V) + 0_W = T(0_V)$, $T(0_V) = 0_W$.

Kernel (Nullspace): Let $T : V \to W$ be a linear transformation. Write down a definition of the kernel of T.

$$\ker(T) = \{ v \in V \mid T(v) = 0_W \}$$

Nullity: The dimension of the kernel of T, $\dim(\ker(T))$, is called the nullity of T.

Image (Range): Let $T : V \to W$ be a linear transformation. Write down a definition of the image of T.

$$\operatorname{Im}(T) = \{ w \in W \mid w = T(v) \text{ for some } v \in V \}$$

Rank: The dimension of the image of T, $\dim(\operatorname{Im}(T))$, is called the rank of T.

Matrices: Let $A \in M_{m \times n}(\mathbb{R})$ be an $m \times n$ matrix. Then $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(x) = Ax$ is linear. In this case, the image of T is the span of the columns of A, and so is called the column space of A.

Ex. 4. Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 5 & 5 & 7 & 7 & 7 \end{bmatrix}$.

(a) Find a basis for the column space of A.
(b) Find a basis for the null space (or kernel) of A.
(c) Find the general solution of the equation $A\vec{x} = \vec{0}$.

We begin by row-reducing A.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 5 & 5 & 7 & 7 & 7 \end{bmatrix} \xrightarrow[5R1-R3 \to R3]{3R1-R2 \to R2} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 & -2 \end{bmatrix} \xrightarrow[-\frac{1}{2}R3]{R2 \leftrightarrow R3} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R1-R2 \to R1} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) There are pivots in columns 1 and 3, so a basis of the column space is formed by the 1st & 3rd columns of the matrix, i.e. a basis of column space is $\left\{ \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} \right\}$.

b) Setting $T(x) = Ax = \vec{0}$ leads to the augmented matrix $\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

using our row-reduction from above. This corresponds to the equations

$$\begin{cases} x_1 + x_2 = 0 \\ x_3 + x_4 + x_5 = 0 \end{cases} \Longrightarrow \begin{array}{l} x_1 = -x_2 \\ x_3 = -x_4 - x_5 \\ x_2, x_4, x_5 \text{ free.} \end{array}$$

Then $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \\ -x_4 - x_5 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

Thus, a basis for the null space of A is $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

c) By part b, the general solution of $A\vec{x} = \vec{0}$ is $x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$ for $x_2, x_4, x_5 \in \mathbb{R}$.

One-to-one: Let $T : V \to W$. Write down what it means to say that T is one-to-one (injective).

If $T(v_1) = T(v_2)$ for some $v_1, v_2 \in V$, then $v_1 = v_2$.

Onto: Let $T : V \to W$. Write down what it means to say that T is onto (surjective).

For all $w \in W$, there exists a $v \in V$ such that $T(v) = w$. (or $\operatorname{Im}(T) = W$)

Isomorphism: A linear transformation that is both one-to-one and onto is called an isomorphism.

Ex. 5. Let V and W be vector spaces and let T be a linear transformation from V to W. Prove that T is one-to-one if and only if $\ker(T) = \{\vec{0}_V\}$. (Note: This is a commonly used result, which you should remember.)

$\Rightarrow$ First suppose that T is one-to-one. Since T is linear, we already know that $0_V \in \ker T$ and so $\{0_V\} \subseteq \ker(T)$. For the reverse containment, let $x \in \ker(T)$. Then $T(x) = 0_W$. But $T(0_V) = 0_W = T(x)$ and since T is one-to-one, we must have $x = 0_V$. Thus, $\ker(T) \subseteq \{0_V\}$. So, $\ker(T) = \{\vec{0}_V\}$.

$\Leftarrow$ Now suppose that $\ker(T) = \{\vec{0}_V\}$. We want to show that T is one-to-one, so suppose that $T(v_1) = T(v_2)$ for $v_1, v_2 \in V$. Adding $-T(v_2)$ to both sides, we have $T(v_1) + -T(v_2) = 0_W$ and since T is linear, $T(v_1 + -v_2) = 0_W$. Then $v_1 + -v_2 \in \ker(T) = \{0_V\}$. Thus, $v_1 + -v_2 = 0_V$ which implies that $v_1 = v$. Thus, T is one-to-one.

Ex. 6. Let $T : V \to V$ be a linear transformation on a finite dimensional vector space V. Suppose that T is one-to-one. Prove that if $\{v_1, v_2, \ldots v_n\}$ is a basis for V then $\{T(v_1), T(v_2), \ldots T(v_n)\}$ is also a basis for V.

Suppose that $\{v_1, \ldots, v_n\}$ is a basis for V. We want to show that $\{T(v_1), \ldots, T(v_n)\}$ is also a basis for V, so we first show that this set is linearly independent. Let $a_1, \ldots, a_n \in \mathbb{R}$ such that $a_1T(v_1) + \ldots + a_nT(v_n) = 0$. Since T is linear, $T(a_1v_1 + \ldots + a_nv_n) = 0$ and so $a_1v_1 + \ldots + a_nv_n \in \ker(T)$. And since T is one-to-one, $\ker(T) = \{0\}$ so $a_1v_1 + \ldots + a_nv_n = 0$. But $\{v_1, \ldots, v_n\}$ is linearly independent, so we must have $a_1 = \ldots = a_n = 0$. Hence, $\{T(v_1), \ldots, T(v_n)\}$ is linearly independent as well. Now since $\{v_1, \ldots, v_n\}$ is a basis for V, $\dim(V) = n$. Then since $\{T(v_1), \ldots, T(v_n)\}$ also has n elements and is linearly independent, this set must span V as well. Thus, $\{T(v_1), \ldots, T(v_n)\}$ is a basis of V.

Ex. 7. Let V be a vector space and $S = \{v_1, v_2\}$ be a subset of V. Define $T : \mathbb{R}^2 \to V$ by

$$T(x_1, x_2) = 2x_1v_1 + 3x_2v_2.$$

(a) Prove that T is linear.
(b) Prove that if S is linearly independent then T is one-to-one.
(c) Prove that if $\mathrm{Span}(S) = V$ then T is onto.

a) Let $c \in \mathbb{R}$ and $x = (x_1, x_2), y = (y_1, y_2)$ be in $\mathbb{R}^2$.

$T(cx+y) = T(cx_1+y_1, cx_2+y_2) = 2(cx_1+y_1)v_1 + 3(cx_2+y_2)v_2$

$= c(2x_1v_1 + 3x_2v_2) + (2y_1v_1 + 3y_2v_2) = cT(x) + T(y).$

Thus, T is linear.

b) Suppose that S is linearly independent. We want to show that T is one-to-one, so let $T(x) = T(y)$ for some $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$.

Then $2x_1v_1 + 3x_2v_2 = 2y_1v_1 + 3y_2v_2$ and subtracting gives $(2x_1 - 2y_1)v_1 + (3x_2 - 3y_2)v_2 = 0$. Since $\{v_1, v_2\}$ is linearly independent, we must have $2x_1 - 2y_1 = 0$ and $3x_2 - 3y_2 = 0$. Thus, $x_1 = y_1$ and $x_2 = y_2$.

So $x = y$ and T is one-to-one.

c) Suppose that $\mathrm{Span}(S) = V$. We want to show that T is onto, so let $v \in V$. We want to find an $x \in \mathbb{R}^2$ s.t. $T(x) = v$

Scratchwork: $T(x) = 2x_1v_1 + 3x_2v_2 = v$

$\mathrm{Span}(S) = V \implies v = c_1v_1 + c_2v_2$ for some $c_1, c_2 \in \mathbb{R}$

So we need $2x_1 = c_1$ and $3x_2 = c_2$

Then set $x_1 = \frac{1}{2}c_1$ and $x_2 = \frac{1}{3}c_2$.

Since $\mathrm{Span}(S) = V$, we can write $v = c_1v_1 + c_2v_2$ for some $c_1, c_2 \in \mathbb{R}$.

Let $x_1 = \frac{1}{2}c_1$ and $x_2 = \frac{1}{3}c_2$. Then for $x = (x_1, x_2)$ we have

$T(x) = 2(\frac{1}{2}c_1)v_1 + 3(\frac{1}{3}c_2)v_2 = c_1v_1 + c_2v_2 = v.$

Thus, T is onto.

One-to-one, Onto, and Dimension: Let $T : V \to W$ be linear.

- T is one-to-one if and only if $\text{nullity}(T) = 0$. (Ex. 5)
- If $\dim(W) < \infty$, T is onto if and only if $\text{rank}(T) = \dim(W)$.

Rank-Nullity Theorem (Dimension Theorem): $\text{rank}(T) + \text{nullity}(T) = \dim(V)$

Challenge: Explain how the following "two-out-of-three theorem" follows from the Rank-Nullity Theorem and the two results above.

Two-out-of-three Theorem for Linear Transformations: Let $T : V \to W$ be a linear transformation, and suppose that at least one of V, W is finite-dimensional. If any two of the following conditions hold, then all three hold (and so T is an isomorphism).

(1) T is one-to-one.
(2) T is onto.
(3) $\dim(V) = \dim(W)$.

1 & 2 $\Rightarrow$ $\text{nullity}(T) = 0$, $\text{rank}(T) = \dim(W)$ $\Rightarrow$ $\dim(W) + 0 = \dim(V)$

1 & 3 $\Rightarrow$ $\text{rank}(T) + 0 = \dim(V) = \dim(W)$

2 & 3 $\Rightarrow$ $\dim(W) + \text{nullity}(T) = \dim(W)$

Ex. 8. Let A be a 4×6 matrix with real coefficients.

(a) Can the columns of A be linearly independent?
(b) Prove that the nullity of A is at least 2.
(c) Does the equation $A\vec{x} = \vec{0}$ have a unique solution?
(d) Assume that the nullity of A is exactly 2. What does this imply about the rank of A?

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{16} \\ a_{21} & \ddots & & \\ a_{31} & & \ddots & \\ a_{41} & & & a_{46} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_6 \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_4 \end{bmatrix}$$

a) The columns of A form a set of six vectors in $\mathbb{R}^4$. Since $\dim(\mathbb{R}^4) = 4$, this set cannot be linearly independent.

b) Since A is 4×6, A maps $V = \mathbb{R}^6$ into $W = \mathbb{R}^4$.
Then from the rank-nullity theorem, $\text{rank}(A) + \text{nullity}(A) = 6$.
Since the $\text{Im}(A)$ is a subspace of $\mathbb{R}^4$, the possible values for $\text{rank}(A)$ are $0, 1, 2, 3$, or 4. Hence, the possible values for $\text{nullity}(A)$ are $6, 5, 4, 3$, or 2. So, the nullity is at least 2.

c) Since the nullity is at least 2, the kernel of A is not trivial. Thus, the equation $A\vec{x} = \vec{0}$ does not have a unique solution.

d) If $\text{nullity}(A) = 2$ then $\text{rank}(A) = 4$.

Ex. 9. Suppose that $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map such that $\|T(\mathbf{x})\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$. Prove that T is an isomorphism.

We first show that T is one-to-one by showing $\ker(T) = \{\vec{0}\}$. Let $x \in \ker(T)$. Then $T(x) = \vec{0}$ and so $\|T(x)\| = \|\vec{0}\| = 0$. Then since $\|T(x)\| = \|x\|$, we have $\|x\| = 0$. Hence, $x = \vec{0}$. So, $\ker(T) = \{\vec{0}\}$ and T is one-to-one. And since $T: \mathbb{R}^n \to \mathbb{R}^n$ and clearly $\dim(\mathbb{R}^n) = \dim(\mathbb{R}^n) = n$, T must also be onto. Thus, T is an isomorphism.

Ex. 10. Let $P_2 = \{a + bt + ct^2 : a, b, c \in \mathbb{R}\}$ and $T : P_2 \to \mathbb{R}^2$ be defined by $T(p) = \begin{bmatrix} p(1) \\ p'(1) \end{bmatrix}$.

(a) Prove that T is linear.
(b) Find a basis for the kernel (or null space) of T.
(c) Find the rank and nullity of T.
(d) Is T one-to-one? Is T onto? Justify your answers.

a) Let $p, q \in P_2$ and $k \in \mathbb{R}$. Then

$$T(kp+q) = \begin{bmatrix} (kp+q)(1) \\ (kp+q)'(1) \end{bmatrix} = \begin{bmatrix} kp(1)+q(1) \\ kp'(1)+q'(1) \end{bmatrix} = k\begin{bmatrix} p(1) \\ p'(1) \end{bmatrix} + \begin{bmatrix} q(1) \\ q'(1) \end{bmatrix} = kT(p) + T(q).$$

Thus, T is linear.

b) Let $p(t) = a + bt + ct^2 \in \ker(T)$. Then $T(p) = \begin{bmatrix} p(1) \\ p'(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and so $p(1) = a+b+c = 0$, $p'(1) = b + 2c = 0$.

$$\begin{cases} a+b+c = 0 \\ b+2c = 0 \end{cases} \xrightarrow{E1-E2 \to E1} \begin{cases} a \quad -c = 0 \\ b + 2c = 0 \end{cases} \Rightarrow \begin{matrix} a = c \\ b = -2c \\ c \text{ free} \end{matrix}$$

Then $p(t) = c + -2ct + ct^2 = c(1 - 2t + t^2)$. So $\{1 - 2t + t^2\}$ is a basis of $\ker(T)$.

c) By part b, $\text{nullity}(T) = 1$ and since $\dim(P_2) = 3$, $\text{rank}(T) = 2$ by the rank-nullity theorem.

d) Since $\text{nullity}(T) = 1 \neq 0$, T is not one-to-one. Since $\dim(\mathbb{R}^2) = 2 = \text{rank}(T)$, T is onto.

Additional Problems

Ex. 11. Let V be a vector space. Suppose that $\{u, v, w\}$ is a basis of V. Is the set $\{u+2v, v+2w, u+2w\}$ also a basis of V? Justify your answer. Use the strategy from Ex. 1

Ex. 12. Let V_1 and V_2 be vector spaces and let $T: V_1 \to V_2$ be a linear transformation. For $W_2 \subseteq V_2$, let $W_1 = \{v \in V_1 : T(v) \in W_2\}$. Show that if W_2 is a subspace of V_2 then W_1 is a subspace of V_1. Similar to Ex. 13 b.

Ex. 13. Let $T: V \to W$ be a linear transformation between vector spaces V and W. Prove the following.

(a) The kernel of T is a subspace of V. see Prop 2.3.2 in text for a proof

(b) The image of T is a subspace of W. " 2.3.11 "

Ex. 14. Let $A = \begin{bmatrix} 1 & -1 & 1 & 3 \\ -1 & 1 & 0 & -2 \\ 2 & -2 & 4 & 11 \end{bmatrix}$. Find bases for the column space and null space of A. Similar to Ex. 4

Ex. 15. Let $T: \mathbb{R}^3 \to P_2(\mathbb{R})$ be defined by $T(a_1, a_2, a_3) = (a_1 - a_2) + a_2 x + (a_1 + a_3)x^2$. Prove that T is an isomorphism. Note: $\dim(\mathbb{R}^3) = \dim(P_2)$ so only need to show 1-1

Ex. 16. Let V and W be vector spaces and let T be a linear transformation from V to W. Suppose that T is one-to-one and $\{v_1, v_2, \ldots v_n\}$ is a set of n linearly independent vectors in V. Prove that $\{T(v_1), T(v_2), \ldots T(v_n)\}$ is linearly independent in W. Similar to Ex. 6

Ex. 17. Let V and W be vector spaces and $T: V \to W$ be a linear transformation. Suppose that $\{T(v_1), T(v_2), \ldots T(v_n)\}$ is linearly independent.

(a) Prove that $\{v_1, v_2, \ldots v_n\}$ is linearly independent. Start with $a_1v_1 + \ldots + a_nv_n = 0$ and take T of both sides

(b) Suppose in addition that $\text{Span}(\{v_1, v_2, \ldots v_n\}) = V$. Prove that T is one-to-one. Let $x = a_1v_1 + \ldots + a_nv_n \in \ker T$ and show $x = 0$.

Ex. 18. Let V and W be vector spaces and $T: V \to W$ be a linear transformation. For $c \in \mathbb{R}$ with $c \neq 0$, define a map $S: V \to W$ by $S(v) = cT(v)$. Prove that if T is onto then S is onto as well. Let $w \in W$ and $v \in V$ s.t. $T(v) = w$. Then $S(\frac{1}{c}v) = w$.

Ex. 19. Let $V = M_{2\times 2}(\mathbb{R})$ be the vector space of 2×2 matrices with real coefficients and $W = P_2(\mathbb{R})$ be the vector space of polynomials of degree less than or equal to 2. Suppose that $T: V \to W$ is a linear transformation. $\dim(V) = 2 \cdot 2 = 4$ $\dim(W) = 3$

$4 = \text{null}(T) + \underset{\max = 3}{\text{rank}(T)}$

(a) Explain what is meant by the kernel, or null space, of T.

(b) What are the possible values of the nullity, or the dimension of the kernel, of T? Justify your answer. 1, 2, 3, or 4

(c) Can T be one-to-one? Can T be onto?

no yes

Ex. 20. Let $T: V \to W$ be an isomorphism between finite dimensional vector spaces V and W. Let U be a subspace of V and let $T(U) = \{T(u) : u \in U\}$. Prove that $\dim(U) = \dim(T(U))$.

Let $\{u_1, \ldots, u_n\}$ be a basis for U and show that $\{T(u_1), \ldots, T(u_n)\}$ is a basis for $T(U)$