1. Addendum: More on Basis and Dimension

(This material appears in Section 2 of the Study Guide, but in Chapter 1 of the text.)

Theorems about dimension from Chapter 1

- If $X \subseteq V$ is a subspace, then $\dim(X) \leq \dim(V)$. Moreover, if $dim(V) < \infty$, then $\dim(X) = \dim(V)$ if and only if $X = V$.
- Let V be a vector space with $\dim(V) = n$, and let $S \subseteq V$ be a set of m distinct vectors in V.
  - If $m < n$, then S cannot span V.
  - If $m > n$, then S cannot be linearly independent.

Challenge: Explain how the following "two-out-of-three theorem" follows from the results above.

Two-out-of-three Theorem for Sets: Let V be a vector space, and let $S \subseteq V$ be a set of n distinct vectors in V. If any two of the following conditions hold, then all three hold (and S is a basis for V).

(1) S is linearly independent.
(2) S spans V.
(3) $\dim(V) = n$.

Ex. 1. Let V be a vector space. Is the set $\{x, 1+x, -1-2x+x^2\}$ a basis of $P_2(\mathbb{R})$? Justify your answer.

2. Linear Transformations

(References: Comps Study Guide for Linear Algebra Section 2;
Damiano & Little, *A Course in Linear Algebra*, Chapter 2)

Linear Transformations: Let V and W be vector spaces and $T : V \to W$. Write down the condition(s) that T must satisfy in order to be linear.

Ex. 2. Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. Suppose that $T(1,2) = (3,4)$ and $T(2,3) = (1,1)$. Find $T(x,y)$ for $(x,y) \in \mathbb{R}^2$.

Ex. 3. Let $T : V \to W$ be a linear transformation between vector spaces V and W. Prove that $T(\vec{0}_V) = \vec{0}_W$. (Note: This is a very commonly used result, which you should remember.)

Kernel (Nullspace): Let $T : V \to W$ be a linear transformation. Write down a definition of the kernel of T.

Nullity: The dimension of the kernel of T, $\dim(\ker(T))$, is called the nullity of T.

Image (Range): Let $T : V \to W$ be a linear transformation. Write down a definition of the image of T.

Rank: The dimension of the image of T, $\dim(\operatorname{Im}(T))$, is called the rank of T.

Matrices: Let $A \in M_{m\times n}(\mathbb{R})$ be an $m \times n$ matrix. Then $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(x) = Ax$ is linear. In this case, the image of T is the span of the columns of A, and so is called the column space of A.

Ex. 4. Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 5 & 5 & 7 & 7 & 7 \end{bmatrix}$.

(a) Find a basis for the column space of A.
(b) Find a basis for the null space (or kernel) of A.
(c) Find the general solution of the equation $A\vec{x} = \vec{0}$.

One-to-one: Let $T : V \to W$. Write down what it means to say that T is one-to-one (injective).

Onto: Let $T : V \to W$. Write down what it means to say that T is onto (surjective).

Isomorphism: A linear transformation that is both one-to-one and onto is called an isomorphism.

Ex. 5. Let V and W be vector spaces and let T be a linear transformation from V to W. Prove that T is one-to-one if and only if $\ker(T) = \{\vec{0}_V\}$. (Note: This is a commonly used result, which you should remember.)

Ex. 6. Let $T : V \to V$ be a linear transformation on a finite dimensional vector space V. Suppose that T is one-to-one. Prove that if $\{v_1, v_2, \ldots v_n\}$ is a basis for V then $\{T(v_1), T(v_2), \ldots T(v_n)\}$ is also a basis for V.

Ex. 7. Let V be a vector space and $S = \{v_1, v_2\}$ be a subset of V. Define $T : \mathbb{R}^2 \to V$ by

$$T(x_1, x_2) = 2x_1v_1 + 3x_2v_2.$$

(a) Prove that T is linear.
(b) Prove that if S is linearly independent then T is one-to-one.
(c) Prove that if $\text{Span}(S) = V$ then T is onto.

One-to-one, Onto, and Dimension: Let $T : V \to W$ be linear.

- T is one-to-one if and only if $\text{nullity}(T) = 0$.
- If $\dim(W) < \infty$, T is onto if and only if $\text{rank}(T) = \dim(W)$.

Rank-Nullity Theorem (Dimension Theorem): $\text{rank}(T) + \text{nullity}(T) = \dim(V)$

Challenge: Explain how the following "two-out-of-three theorem" follows from the Rank-Nullity Theorem and the two results above.

Two-out-of-three Theorem for Linear Transformations: Let $T : V \to W$ be a linear transformation, and suppose that at least one of V, W is finite-dimensional. If any two of the following conditions hold, then all three hold (and so T is an isomorphism).

(1) T is one-to-one.
(2) T is onto.
(3) $\dim(V) = \dim(W)$.

Ex. 8. Let A be a 4×6 matrix with real coefficients.

(a) Can the columns of A be linearly independent?
(b) Prove that the nullity of A is at least 2.
(c) Does the equation $A\vec{x} = \vec{0}$ have a unique solution?
(d) Assume that the nullity of A is exactly 2. What does this imply about the rank of A?

Ex. 9. Suppose that $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map such that $\|T(\mathbf{x})\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$. Prove that T is an isomorphism.

Ex. 10. Let $P_2 = \{a + bt + ct^2 : a, b, c \in \mathbb{R}\}$ and $T : P_2 \to \mathbb{R}^2$ be defined by $T(p) = \begin{bmatrix} p(1) \\ p'(1) \end{bmatrix}$.

(a) Prove that T is linear.
(b) Find a basis for the kernel (or null space) of T.
(c) Find the rank and nullity of T.
(d) Is T one-to-one? Is T onto? Justify your answers.

Additional Problems

Ex. 11. Let V be a vector space. Suppose that $\{u, v, w\}$ is a basis of V. Is the set $\{u+2v, v+2w, u+2w\}$ also a basis of V? Justify your answer.

Ex. 12. Let V_1 and V_2 be vector spaces and let $T : V_1 \to V_2$ be a linear transformation. For $W_2 \subseteq V_2$, let $W_1 = \{v \in V_1 : T(v) \in W_2\}$. Show that if W_2 is a subspace of V_2 then W_1 is a subspace of V_1.

Ex. 13. Let $T : V \to W$ be a linear transformation between vector spaces V and W. Prove the following.
(a) The kernel of T is a subspace of V.
(b) The image of T is a subspace of W.

Ex. 14. Let $A = \begin{bmatrix} 1 & -1 & 1 & 3 \\ -1 & 1 & 0 & -2 \\ 2 & -2 & 4 & 11 \end{bmatrix}$. Find bases for the column space and null space of A.

Ex. 15. Let $T : \mathbb{R}^3 \to P_2(\mathbb{R})$ be defined by $T(a_1, a_2, a_3) = (a_1 - a_2) + a_2 x + (a_1 + a_3)x^2$. Prove that T is an isomorphism.

Ex. 16. Let V and W be vector spaces and let T be a linear transformation from V to W. Suppose that T is one-to-one and $\{v_1, v_2, \ldots v_n\}$ is a set of n linearly independent vectors in V. Prove that $\{T(v_1), T(v_2), \ldots T(v_n)\}$ is linearly independent in W.

Ex. 17. Let V and W be vector spaces and $T : V \to W$ be a linear transformation. Suppose that $\{T(v_1), T(v_2), \ldots T(v_n)\}$ is linearly independent.
(a) Prove that $\{v_1, v_2, \ldots v_n\}$ is linearly independent.
(b) Suppose in addition that $\text{Span}(\{v_1, v_2, \ldots v_n\}) = V$. Prove that T is one-to-one.

Ex. 18. Let V and W be vector spaces and $T : V \to W$ be a linear transformation. For $c \in \mathbb{R}$ with $c \neq 0$, define a map $S : V \to W$ by $S(v) = cT(v)$. Prove that if T is onto then S is onto as well.

Ex. 19. Let $V = M_{2\times 2}(\mathbb{R})$ be the vector space of 2×2 matrices with real coefficients and $W = P_2(\mathbb{R})$ be the vector space of polynomials of degree less than or equal to 2. Suppose that $T : V \to W$ is a linear transformation.
(a) Explain what is meant by the kernel, or null space, of T.
(b) What are the possible values of the nullity, or the dimension of the kernel, of T? Justify your answer.
(c) Can T be one-to-one? Can T be onto?

Ex. 20. Let $T : V \to W$ be an isomorphism between finite dimensional vector spaces V and W. Let U be a subspace of V and let $T(U) = \{T(u) \ : \ u \in U\}$. Prove that $\dim(U) = \dim(T(U))$.