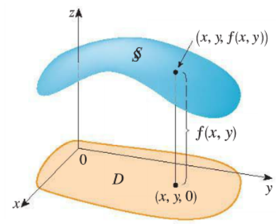
2. Functions of Several Variables
(from Stewart, *Calculus*, Chapter 14)



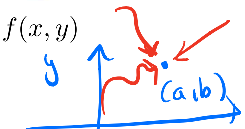
Graph of $f(x, y)$

Functions of Two Variables: $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$. The graph of the function $f(x, y)$ is the set of all points $(x, y, z) \in \mathbb{R}^3$ such that $(x, y) \in D$ and $z = f(x, y)$. The graph of a function of two variables is a surface in $\mathbb{R}^3$.

Functions of Three Variables: $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^3$. The graph of a function $f(x, y, z)$ of three variables would lie in $\mathbb{R}^4$. For a visual representation, we often look instead at the level surfaces of f, i.e. the surfaces in $\mathbb{R}^3$ with equations $f(x, y, z) = k$ for constants k.

Continuity: A function $f(x, y)$ is continuous at (a, b) if $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

Note: the value must be the same as $(x,y) \to (a,b)$ along any possible path.



Partial Derivatives

$$f_x(x, y) = \frac{\partial f}{\partial x}(x, y) = \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h} \qquad \text{and} \qquad f_y = \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

Shortcut for Finding Partial Derivatives

- To find f_x, regard y as constant and differentiate $f(x, y)$ with respect to x.
- To find f_y, regard x as constant and differentiate $f(x, y)$ with respect to y.

Ex. 1. Consider the function $f(x, y) = \begin{cases} \frac{4x^3}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$

(a) Show that f is continuous at $(0, 0)$.

First we change to polar coordinates. Let $x = r\cos\theta$, $y = r\sin\theta$. Then $x^2+y^2 = r^2$ and $\frac{4x^3}{x^2+y^2} = \frac{4r^3\cos^3\theta}{r^2} = 4r\cos^3\theta$. Now $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0^+} 4r\cos^3\theta$. Since $-1 \le \cos\theta \le 1$, $-4r \le 4r\cos^3\theta \le 4r$ for $r > 0$. And $\lim_{r\to 0^+}(-4r) = 0 = \lim_{r\to 0^+}(4r)$ so by the Squeeze Theorem, $\lim_{r\to 0^+} 4r\cos^3\theta = 0$ as well. Thus, $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$ and so f is continuous at $(0,0)$.

(b) Find $f_x(0, 0)$ and $f_y(0, 0)$.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{4h^3}{h^2} - 0}{h} = \lim_{h\to 0} \frac{4h^3}{h^3} = 4.$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{0}{h^2} - 0}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

Ex. 2. Let $f(x,y) = \begin{cases} \frac{3x^2y^2}{2x^4+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

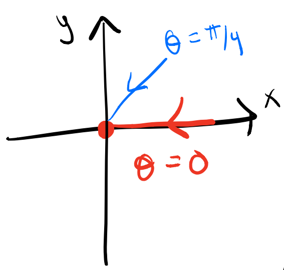
(a) Is f continuous at $(0,0)$? Justify your answer.

In polar coordinates, $f(r\cos\theta, r\sin\theta) = \dfrac{3r^4\cos^2\theta\sin^2\theta}{2r^4\cos^4\theta + r^4\sin^4\theta} = \dfrac{3\cos^2\theta\sin^2\theta}{2\cos^4\theta + \sin^4\theta}$

So $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{r\to 0^+} \dfrac{3\cos^2\theta\sin^2\theta}{2\cos^4\theta+\sin^4\theta}$ is dependent on the angle θ

For example, if we approach along the positive x-axis, we have

$$\lim_{\substack{(x,y)\to(0,0)\\ x>0,\ y=0}} f(x,y) = \lim_{\substack{r\to 0^+\\ \theta=0}} \frac{3\cos^2\theta\sin^2\theta}{2\cos^4\theta+\sin^4\theta} = \lim_{r\to 0^+} \frac{0}{2} = 0$$



On the other hand, approaching along the line $y=x$ for $x>0$,

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x,\ x>0}} f(x,y) = \lim_{\substack{r\to 0^+\\ \theta=\frac{\pi}{4}}} \frac{3\cos^2\theta\sin^2\theta}{2\cos^4\theta+\sin^4\theta} = \frac{3\left(\frac{1}{\sqrt{2}}\right)^2\left(\frac{1}{\sqrt{2}}\right)^2}{2\left(\frac{1}{\sqrt{2}}\right)^4+\left(\frac{1}{\sqrt{2}}\right)^4} = \frac{\frac{3}{4}}{3/4} = 1$$

Since we get different values along different paths, the limit $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist and hence f is not continuous at $(0,0)$.

(b) Find $f_x(0,0)$ and $f_y(0,0)$.

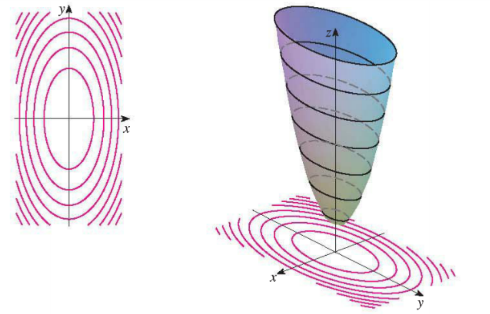
$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{3h^2\cdot 0}{2h^4+0} - 0}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{3\cdot 0\cdot h^2}{2\cdot 0 + h^4} - 0}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

Level Curves and Level Surfaces

Def. The <u>level curves</u> of a function f of two variables are the curves in the xy-plane with equation $f(x, y) = k$ for any constant k. Level curves are the projection onto the xy-plane of traces of the surface in the planes $z = k$. (A contour map is a collection of level curves.)

Def. The <u>level surfaces</u> of a function $f(x, y, z)$ are the surfaces with equations $f(x, y, z) = k$.

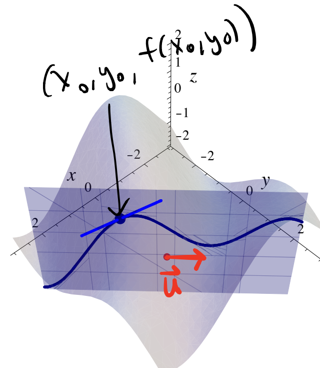


(a) Contour map (b) Horizontal traces are raised level curves

Directional Derivatives and the Gradient Vector

<u>Directional Derivatives</u> $\|\vec{u}\| = 1$

Given a unit vector $\vec{u} = \langle a, b\rangle$, the vertical plane through (x_0, y_0) in the direction of $\vec{u}$ intersects S in a curve C. The slope of the tangent line to C at the point (x_0, y_0) gives the rate of change of f at (x_0, y_0) in the direction of the unit vector $\vec{u}$. This is called the <u>directional derivative</u> of f in the direction of $\vec{u}$, and is denoted $D_{\vec{u}}f(x_0, y_0)$.



Wolfram Demonstrations Project: "Directional Derivatives in 3D"

Note that the partial derivatives f_x and f_y are the directional derivatives of f in the directions of $\vec{i}$ and $\vec{j}$, respectively.

Thm. If f is a differentiable function then f has a directional derivative in the direction of any unit vector $\vec{u}$ and

$$D_{\vec{u}}f = \nabla f \cdot \vec{u}$$

<u>The Gradient Vector</u>

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \qquad \text{or} \qquad \nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

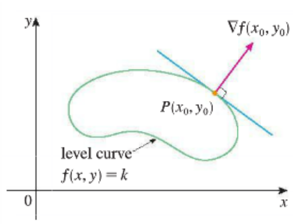
<u>Properties of the Gradient</u>: Let θ be the angle between a unit vector $\vec{u}$ and ∇f at a point P.

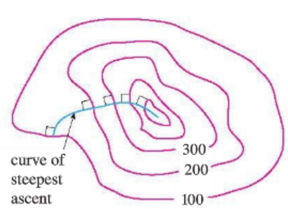
$$D_{\vec{u}}f = \nabla f \cdot \vec{u} = \|\nabla f\| \cos(\theta)$$

$\|\nabla f\| = \sqrt{(f_x)^2 + (f_y)^2}$

1. The maximum value of the directional derivative $D_{\vec{u}}f(P)$ at a point P is $\|\nabla f(P)\|$ and it occurs when $\vec{u}$ has the same direction as the gradient vector $\nabla f(P)$. So $\nabla f(P)$ points in the direction of maximum rate of increase of f at P.

2. $\nabla f(x, y)$ is perpendicular to the level curves of a function $f(x, y)$. Similarly, $\nabla f(x, y, z)$ is perpendicular to the level surfaces of a function $f(x, y, z)$.





follows the gradient at each pt.

Ex. 3. Find the directional derivative of the function $f(x,y,z) = y^2 e^{xyz}$ at the point $(0,1,-1)$ in the direction of the vector $\langle 4,2,1\rangle = \vec{v}$

$\|\vec{v}\| = \sqrt{4^2+2^2+1^2} = \sqrt{21}$ so the unit vector is $\vec{u} = \frac{1}{\sqrt{21}}\langle 4,2,1\rangle$

$$\nabla f = \langle y^2 yz e^{xyz},\ 2ye^{xyz} + y^2 xze^{xyz},\ y^2 xy e^{xyz}\rangle$$

$$\nabla f(0,1,-1) = \langle -1, 2, 0\rangle$$

$$D_{\vec{u}} f(0,1,-1) = \langle -1,2,0\rangle \cdot \frac{1}{\sqrt{21}}\langle 4,2,1\rangle = \frac{1}{\sqrt{21}}(-4+4+0) = 0$$

Ex. 4. Let $f(x,y) = 2x^2 + xy^2$. Find a unit vector that points in the direction of the maximum rate of increase at the point $(1,2)$. What is the rate of change of f in this direction?

The max rate of increase is in the direction of the gradient at $(1,2)$.

$\nabla f(x,y) = \langle 4x+y^2,\ 2xy\rangle$ and $\nabla f(1,2) = \langle 8,4\rangle$, so

$\|\nabla f(1,2)\| = \sqrt{8^2+4^2} = \sqrt{80} = 4\sqrt{5}$. Then $\vec{u} = \frac{1}{4\sqrt{5}}\langle 8,4\rangle = \frac{1}{\sqrt{5}}\langle 2,1\rangle$

is a unit vector pointing in the direction of max. increase.

And the rate of increase of f in this direction is $\|\nabla f(1,2)\| = 4\sqrt{5}$.

Ex. 5. The temperature at the point (x,y,z) is $T(x,y,z) = \frac{1}{\pi}\sin(\pi xy) + \ln(z^2+1) + 60$.

(a) Let $\vec{v} = -\vec{i} + 2\vec{j} + 2\vec{k}$. What is the rate of change of the temperature at the point $(2,-1,1)$ in the direction of $\vec{v}$?

$\|\vec{v}\| = \sqrt{1+4+4} = 3$ so $\vec{u} = \frac{1}{3}\langle -1,2,2\rangle$.

$$\nabla T = \langle \tfrac{1}{\pi}\cdot \pi y\cos(\pi xy),\ x\cos(\pi xy),\ \tfrac{1}{z^2+1}\cdot 2z\rangle$$

$$\nabla T(2,-1,1) = \langle -\cos(-2\pi),\ 2\cos(-2\pi),\ \tfrac{1}{2}\cdot 2(1)\rangle = \langle -1,2,1\rangle$$

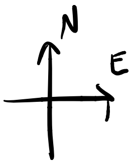
so $D_{\vec{u}} T(2,-1,1) = \frac{1}{3}\langle -1,2,2\rangle \cdot \langle -1,2,1\rangle = \frac{1}{3}(1+4+2) = \frac{7}{3}$.

Thus, the rate of change of T at $(2,-1,1)$ in the direction of $\vec{v}$ is $\frac{7}{3}$.

(b) Find a vector pointing in the direction in which the temperature increases most rapidly at the point $(2, -1, 1)$.

The temperature increases most rapidly in the direction of

$\nabla T(2,-1,1) = \langle -1, 2, 1 \rangle$ from part a.



Ex. 6. A hiker is walking on a mountain path. The surface of the mountain is modeled by $f(x, y) = 1 - 4x^2 - 3y^2$. The positive x-axis points to the East direction and the positive y-axis points North.

(a) Suppose the hiker is now at the point $P(1/4, -1/2, 0)$ and heading North. Is she ascending or descending?

A unit vector pointing north is $\vec{u} = \langle 0, 1 \rangle$.

$\nabla f(x,y) = \langle -8x, -6y \rangle$ so $\nabla f(\frac{1}{4}, -\frac{1}{2}) = \langle -2, 3 \rangle$

$D_{\vec{u}} f(\frac{1}{4}, \frac{-1}{2}) = \langle -2, 3 \rangle \cdot \langle 0, 1 \rangle = 3 > 0$. So the hiker is ascending.

(b) When the hiker is at the point $Q(1/4, 0, 3/4)$, in which direction should she initially head to descend most rapidly?

$D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u} = \|\nabla f(x,y)\| \cos\theta$ where θ is the angle between $\nabla f(x,y)$ and $\vec{u}$. Since $\|\nabla f(x,y)\|$ at $(x,y) = (\frac{1}{4}, 0)$ is fixed, $D_{\vec{u}} f(\frac{1}{4}, 0)$ will take its largest negative value when $\cos\theta = -1$, i.e. $\theta = \pi$. Thus, the hiker will descend most rapidly if she initially heads in the direction opposite to $\nabla f(\frac{1}{4}, 0) = \langle -2, 0 \rangle$.

Thus, she should initially head in the direction of $\langle 2, 0 \rangle$ or east.

Tangent Planes

Tangent Planes: The tangent plane to a surface S at a point $P = (x_0, y_0, z_0)$ is the plane containing the tangent line to any curve C on S passing through P.

Equations of Tangent Planes

If S is the graph of a function, i.e. $z = f(x,y)$ is given explicitly as a function of x and y:

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

If S is the graph of a level surface $F(x,y,z) = k$ and $\nabla F(x_0, y_0, z_0) \neq 0$:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

Rmk. This second equation is just the usual scalar equation of a plane, with $\vec{n} = \nabla F$. This equation can be used in the first case as well. rearranging: $z = f(x,y) \Rightarrow \underbrace{f(x,y) - z}_{F(x,y,z)} = 0$

Ex. 7. Let $f(x,y) = y + \sin(x/y)$. Find an equation of the tangent plane to the graph of $z = f(x,y)$ at the point $(0,3,3)$. $f(0,3) = 3$

$f_x = \frac{1}{y}\cos(x/y)$ $\quad f_y = 1 + \frac{-x}{y^2}\cos(x/y)$

$f_x(0,3) = \frac{1}{3}$ $\quad f_y(0,3) = 1$

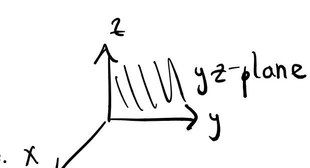
Then an eq. of the tangent plane is $z - 3 = \frac{1}{3}(x - 0) + 1(y - 3)$

or $\frac{1}{3}x + y - z = 0$

Ex. 8. Consider the surface S given by the equation $x^2y - yz^2 + z = 1$.

(a) Find an equation of the tangent plane to S at the point $(11, 0, 1)$.

Let $F(x,y,z) = x^2y - yz^2 + z$. Then $\nabla F(x,y,z) = \langle 2xy, x^2 - z^2, -2yz + 1\rangle$ and $\nabla F(11,0,1) = \langle 0, 120, 1\rangle$. So an eq of the tangent plane is $0(x - 11) + 120(y - 0) + 1(z - 1) = 0$ or $120y + z = 1$.



(b) Find two points on the surface S where the tangent plane at P is parallel to the yz-plane.

If the tangent plane is $\parallel$ to the yz-plane, the normal vector is $\parallel$ to $\langle 1,0,0\rangle$. Thus, we want two points s.t. $x^2y - yz^2 + z = 1$ (*) and $\nabla F(x,y,z) \parallel \langle 1,0,0\rangle$, i.e. $\langle 2xy, x^2 - z^2, -2yz + 1\rangle = \lambda\langle 1,0,0\rangle$ for some $\lambda \in \mathbb{R}$. This gives

$2xy = \lambda$

$x^2 - z^2 = 0 \Rightarrow x^2 = z^2$ and substituting into (*) gives $z^2y - yz^2 + z = 1 \Rightarrow z = 1$.

$-2yz + 1 = 0$

Then from $-2yz + 1 = 0$ we have $-2y = -1 \Rightarrow y = 1/2$.

And $x^2 = z^2 = 1 \Rightarrow x = \pm 1$. Thus, the two points are $(\pm 1, \frac{1}{2}, 1)$.