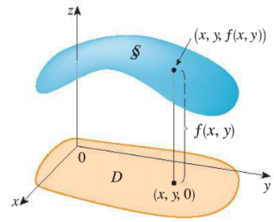
2. Functions of Several Variables

(from Stewart, *Calculus*, Chapter 14)

Functions of Two Variables: $f : D \to \mathbb{R},\ D \subseteq \mathbb{R}^2$. The graph of the function $f(x,y)$ is the set of all points $(x,y,z) \in \mathbb{R}^3$ such that $(x,y) \in D$ and $z = f(x,y)$. The graph of a function of two variables is a surface in $\mathbb{R}^3$.

Functions of Three Variables: $f : D \to \mathbb{R},\ D \subseteq \mathbb{R}^3$. The graph of a function $f(x,y,z)$ of three variables would lie in $\mathbb{R}^4$. For a visual representation, we often look instead at the level surfaces of f, i.e. the surfaces in $\mathbb{R}^3$ with equations $f(x,y,z) = k$ for constants k.



Graph of $f(x,y)$

Continuity: A function $f(x,y)$ is continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

Partial Derivatives

$$f_x(x,y) = \frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \qquad \text{and} \qquad f_y = \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Shortcut for Finding Partial Derivatives

- To find f_x, regard y as constant and differentiate $f(x,y)$ with respect to x.
- To find f_y, regard x as constant and differentiate $f(x,y)$ with respect to y.

Ex. 1. Consider the function $f(x,y) = \begin{cases} \frac{4x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

(a) ~~Show that f is continuous at $(0,0)$.~~ *This is no longer on the Comps syllabus.*

(b) Find $f_x(0,0)$ and $f_y(0,0)$.

Ex. 2. Let $f(x, y) = \begin{cases} \frac{3x^2y^2}{2x^4+y^4} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$

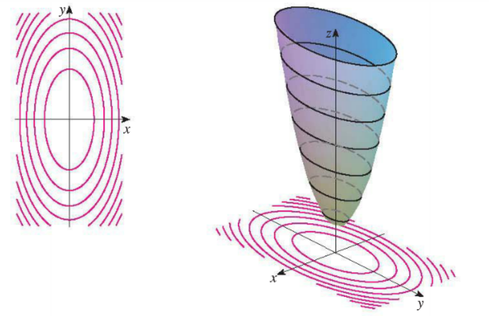
(a) ~~Is f continuous at $(0, 0)$? Justify your answer.~~ *This is no longer on the Comps syllabus.*

(b) Find $f_x(0, 0)$ and $f_y(0, 0)$.

Level Curves and Level Surfaces

Def. The <u>level curves</u> of a function f of two variables are the curves in the xy-plane with equation $f(x, y) = k$ for any constant k. Level curves are the projection onto the xy-plane of traces of the surface in the planes $z = k$. (A contour map is a collection of level curves.)

Def. The <u>level surfaces</u> of a function $f(x, y, z)$ are the surfaces with equations $f(x, y, z) = k$.

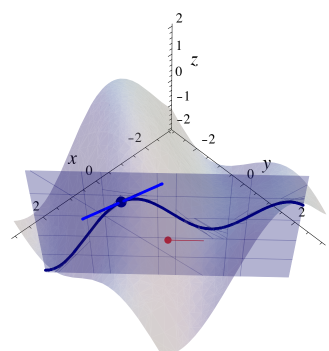


(a) Contour map (b) Horizontal traces are raised level curves

Directional Derivatives and the Gradient Vector

<u>Directional Derivatives</u>

Given a unit vector $\vec{u} = \langle a, b \rangle$, the vertical plane through (x_0, y_0) in the direction of $\vec{u}$ intersects S in a curve C. The slope of the tangent line to C at the point (x_0, y_0) gives the rate of change of f at (x_0, y_0) in the direction of the unit vector $\vec{u}$. This is called the <u>directional derivative</u> of f in the direction of $\vec{u}$, and is denoted $D_{\vec{u}} f(x_0, y_0)$.



Note that the partial derivatives f_x and f_y are the directional derivatives of f in the directions of $\vec{i}$ and $\vec{j}$, respectively.

Thm. If f is a differentiable function then f has a directional derivative in the direction of any unit vector $\vec{u}$ and

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}$$

<u>The Gradient Vector</u>

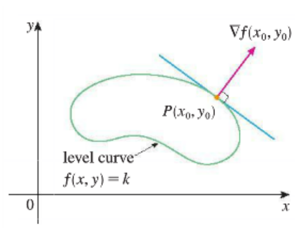
$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \qquad \text{or} \qquad \nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

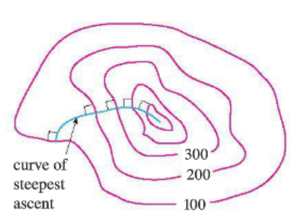
<u>Properties of the Gradient</u>: Let θ be the angle between a unit vector $\vec{u}$ and ∇f at a point P.

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = \|\nabla f\| \cos(\theta)$$

1. The maximum value of the directional derivative $D_{\vec{u}} f(P)$ at a point P is $\|\nabla f(P)\|$ and it occurs when $\vec{u}$ has the same direction as the gradient vector $\nabla f(P)$. So $\nabla f(P)$ points in the direction of maximum rate of increase of f at P.

2. $\nabla f(x, y)$ is perpendicular to the level curves of a function $f(x, y)$. Similarly, $\nabla f(x, y, z)$ is perpendicular to the level surfaces of a function $f(x, y, z)$.





Ex. 3. Find the directional derivative of the function $f(x, y, z) = y^2 e^{xyz}$ at the point $(0, 1, -1)$ in the direction of the vector $\langle 4, 2, 1 \rangle$.

Ex. 4. Let $f(x, y) = 2x^2 + xy^2$. Find a unit vector that points in the direction of the maximum rate of increase at the point $(1, 2)$. What is the rate of change of f in this direction?

Ex. 5. The temperature at the point (x, y, z) is $T(x, y, z) = \frac{1}{\pi}\sin(\pi xy) + \ln(z^2 + 1) + 60$.

(a) Let $\vec{v} = -\vec{i} + 2\vec{j} + 2\vec{k}$. What is the rate of change of the temperature at the point $(2, -1, 1)$ in the direction of $\vec{v}$?

(b) Find a vector pointing in the direction in which the temperature increases most rapidly at the point $(2, -1, 1)$.

Ex. 6. A hiker is walking on a mountain path. The surface of the mountain is modeled by $f(x, y) = 1 - 4x^2 - 3y^2$. The positive x-axis points to the East direction and the positive y-axis points North.

(a) Suppose the hiker is now at the point $P(1/4, -1/2, 0)$ and heading North. Is she ascending or descending?

(b) When the hiker is at the point $Q(1/4, 0, 3/4)$, in which direction should she initially head to descend most rapidly?

Tangent Planes

Tangent Planes: The tangent plane to a surface S at a point $P = (x_0, y_0, z_0)$ is the plane containing the tangent line to any curve C on S passing through P.

Equations of Tangent Planes

If S is the graph of a function, i.e. $z = f(x, y)$ is given explicitly as a function of x and y:

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

If S is the graph of a level surface $F(x, y, z) = k$ and $\nabla F(x_0, y_0, z_0) \neq 0$:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

Rmk. This second equation is just the usual scalar equation of a plane, with $\vec{n} = \nabla F$. This equation can be used in the first case as well.

Ex. 7. Let $f(x, y) = y + \sin(x/y)$. Find an equation of the tangent plane to the graph of $z = f(x, y)$ at the point $(0, 3, 3)$.

Ex. 8. Consider the surface S given by the equation $x^2y - yz^2 + z = 1$.

(a) Find an equation of the tangent plane to S at the point $(11, 0, 1)$.

(b) Find two points on the surface S where the tangent plane at P is parallel to the yz-plane.