Additional Problems

Ex. 7. Let $P_n(\mathbb{R})$ be the vector space of polynomials with real coefficients of degree at most n. Define $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ by $T(f) = \int_0^x f(t)\,dt$.

(a) Compute the matrix of T with respect to the standard bases $\{1, x, x^2\}$ of $P_2(\mathbb{R})$ and $\{1, x, x^2, x^3\}$ of $P_3(\mathbb{R})$.

(b) Is T one-to one? Is T onto?

(c) Find bases for $\ker(T)$ and $\mathrm{Im}(T)$.

a) Let $\alpha = \{1, x, x^2\}$ and $\beta = \{1, x, x^2, x^3\}$.

$$[T]_\alpha^\beta = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$T(1) = \int_0^x 1\,dt = x \rightarrow [T(1)]_\beta = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$T(x) = \int_0^x t\,dt = \frac{1}{2}x^2$

$\vdots$

$T(x^2) = \int_0^x t^2\,dt = \frac{1}{3}x^3$

b) Since $[T]_\alpha^\beta$ has 3 pivots and there are no free variables, nullity$(T) = 0$ and rank$(T) = 3$. Since $\dim(P_3(\mathbb{R})) = 4$, T is one-to-one but not onto.

c) Since T is one-to-one, $\ker(T) = \{\vec{0}\}$ and so a basis the empty set ϕ.

Since the columns of $[T]_\alpha^\beta$ are linearly independent, the polynomials corresponding to these coordinate vectors for a basis for $\mathrm{Im}(T)$. Thus, a basis of $\mathrm{Im}(T)$ is $\{x, \frac{1}{2}x^2, \frac{1}{3}x^3\}$ or $\{x, x^2, x^3\}$.

Ex. 8. Let $M_{2\times 2}(\mathbb{R})$ be the vector space of 2×2 matrices with real coefficients and let $\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ be its standard basis.

(a) Let $T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ by $T(A) = a+b+c+d$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Compute the matrix of T with respect to the bases α for $M_{2\times 2}(\mathbb{R})$ and $\beta = \{1\}$ for $\mathbb{R}$.
(b) Find a basis for $\ker(T)$.
(c) Is T one-to one? Is T onto?

$[1 \ \ 1 \ \ 1 \ \ 1 \ \ 1]$

a) $T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 = T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

So $[T]_\alpha^\beta = [1 \ \ 1 \ \ 1 \ \ 1]$

b) We have $a+b+c+d=0 \Rightarrow a = -b-c-d$ and b, c, d are free.

Then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -b-c-d & b \\ c & d \end{pmatrix} = b\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$,

So a basis for $\ker(T)$ is $\left\{ \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$.

c) Since $\ker(T) \neq \{\vec{0}\}$, T is not one-to-one. And $\dim(\ker(T)) = 3$ while $\dim(M_{2\times 2}(\mathbb{R})) = 4$, so by the rank-nullity theorem, $\dim(\operatorname{Im}(T)) = 1 = \dim(\mathbb{R})$. Thus, T is onto.

Ex. 9. Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be the linear transformation with matrix representation $[T]_{std}^{std} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 5 & 5 & 7 & 7 & 7 \end{bmatrix}$ with respect to the standard bases on $\mathbb{R}^5$ and $\mathbb{R}^3$. Find a basis for $\operatorname{Ker}(T)$ and $\operatorname{Im}(T)$.

Using Ex. 4 from Notes #2, a basis for $\ker(T)$ is $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

and a basis for $\operatorname{Im}(T)$ is $\left\{ \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} \right\}$.

Ex. 10. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (2x + y - 4z, 4y - 5z, -z)$. Determine whether or not T is invertible, and if so, find a formula for $T^{-1}(x, y, z)$.

$$[T]_{std}^{std} = \begin{bmatrix} 2 & 1 & -4 \\ 0 & 4 & -5 \\ 0 & 0 & -1 \end{bmatrix}$$ which has 3 pivot columns and so is invertible.

We compute

$$\left[\begin{array}{ccc|ccc} 2 & 1 & -4 & 1 & 0 & 0 \\ 0 & 4 & -5 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{4R1-R2 \to R1} \left[\begin{array}{ccc|ccc} 8 & 0 & -11 & 4 & -1 & 0 \\ 0 & 4 & -5 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R1 - 11R3 \to R1 \\ R2 - 5R3 \to R2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 8 & 0 & 0 & 4 & -1 & -11 \\ 0 & 4 & 0 & 0 & 1 & -5 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/8 & -11/8 \\ 0 & 1 & 0 & 0 & 1/4 & -5/4 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

So $[T^{-1}]_{std}^{std} = \begin{bmatrix} 1/2 & -1/8 & -11/8 \\ 0 & 1/4 & -5/4 \\ 0 & 0 & -1 \end{bmatrix}$ and $T^{-1}(x, y, z) = \left(\frac{1}{2}x - \frac{1}{8}y - \frac{11}{8}z, \frac{1}{4}y - \frac{5}{4}z, -z\right)$.

Ex. 11. Let A and B be invertible $n \times n$ matrices. Show that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. (Note: This is a common result that you could usually use without proof.)

Let A and B be invertible $n \times n$ matrices. Then A^{-1}, B^{-1} exist and $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$. Similarly, $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$. Thus, AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Ex. 12. Let $\alpha = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and $\beta = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ be bases for a vector space V. Suppose that

$$\begin{aligned}\vec{v}_1 &= \vec{w}_1 - 2\vec{w}_2 - 2\vec{w}_3\\ \vec{v}_2 &= -\vec{w}_2 - \vec{w}_3\\ \vec{v}_3 &= 2\vec{w}_2 - \vec{w}_3.\end{aligned}$$

Compute the change of basis matrix from β to α.

We have $[\vec{v}_1]_\beta = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$, $[\vec{v}_2]_\beta = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$, and $[\vec{v}_3]_\beta = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$. Then

$$[I]_\alpha^\beta = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 2 \\ -2 & -1 & -1 \end{bmatrix}.$$

Computing the inverse, ...

We find $[I]_\beta^\alpha = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1/3 & -2/3 \\ 0 & 1/3 & -1/3 \end{bmatrix}$

Compute the change of basis matrix from β to α.

Ex. 13. Let P_2 be the vector space of polynomials with real coefficients of degree at most 2, and let $\alpha = \{1, x+1, x^2+x+1\}$. It is a fact, which you may assume, that α is a basis for V. Suppose that $T : V \to V$ is a linear transformation and the matrix of T relative to α is $[T]_\alpha^\alpha = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & -1 \\ 2 & 5 & 1 \end{bmatrix}$. Find $T(3x^2+x+2)$.

Note: std $= \{1, x, x^2\}$

$[I]_\alpha^{std} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ so we compute the inverse

$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ $R1-R2 \to R1$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ $R2-R3 \to R2$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ so $[I]_{std}^\alpha = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Then $[3x^2+x+2]_\alpha = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ so

$[T(3x^2+x+2)]_\alpha = [T]_\alpha^\alpha [3x^2+x+2]_\alpha = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & -1 \\ 2 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ -5 \end{bmatrix}$

Thus, $[T(3x^2+x+2)]_{std} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -5 \end{bmatrix}$ and so $T(3x^2+x+2) = 1 - 5x - 5x^2$.

$T(3x^2 + x + 2)$.

Ex. 14. Let $P_2 = \{a + bt + ct^2 : a, b, c \in \mathbb{R}\}$ and $T : P_2 \to \mathbb{R}^2$ be defined by $T(p) = \begin{bmatrix} p(1) \\ p'(1) \end{bmatrix}$. You may assume that T is linear.

(a) Find the matrix representation of T with respect to the bases $\underbrace{\{1, t, t^2\}}_{=\alpha}$ and $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} = \beta$.

(b) Find the rank and nullity of T.

(c) Find bases of the kernel and image of T.

a) $T(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad T(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad T(t^2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad$ So $[T]_\alpha^{std} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

$p(t) = 1 \Rightarrow p(1) = 1$
$p'(t) = 0 \quad p'(1) = 0$

$p(t) = t \quad p(1) = 1$
$p'(t) = 1 \quad p'(1) = 1$

$p(t) = t^2$
$p'(t) = 2t$

$[I]_\beta^{std} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ so $[I]_{std}^\beta = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Then $[T]_\alpha^\beta = [I]_{std}^\beta [T]_\alpha^{std} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$

b) $[T]_\alpha^\beta$ has 2 pivots so $\text{rank}(T) = 2$ and $\text{nullity}(T) = 1$.

c) Since $[T]_\alpha^{std}$ has pivots in the first 2 columns, a basis for $\text{Im}(T)$ is

$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$.

For $p(t) = a + bt + ct^2 \in \ker(T)$ we have $\begin{cases} a + b + c = 0 \\ b + 2c = 0 \end{cases} \Rightarrow \begin{array}{l} a = c \\ b = -2c \\ c \text{ free} \end{array}$

So $p(t) = c + -2c\,t + ct^2 = c(1 - 2t + t^2)$. Then a basis for $\ker(T)$ is $\{1 - 2t + t^2\}$.