3. Matrices

(References: Comps Study Guide for Linear Algebra Section 3;
Damiano & Little, *A Course in Linear Algebra*, Chapters 2 and 3)

Coordinate Vectors: Let V be a vector space and $\alpha = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for V. Then for any $\vec{v} \in V$, there are unique coefficients $a_1, \ldots, a_n \in \mathbb{R}$ such that

$$\vec{v} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n.$$

In this notation, the coordinate vector of $\vec{v}$ with respect to α is $[\vec{v}]_\alpha = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

Matrix of a Linear Transformation: Let $T : V \to W$ be a linear transformation and let $\alpha = \{\vec{v}_1, \ldots, \vec{v}_n\}$ and $\beta = \{\vec{w}_1, \ldots, \vec{w}_n\}$ be bases for V and W, respectively. Then the matrix of T with respect to α and β, denoted $[T]_\alpha^\beta$, is the matrix whose ith column is $[T(\vec{v}_i)]_\beta$, the coordinate vector of $T(\vec{v}_i)$ with respect to β.

$$[T]_\alpha^\beta = \begin{array}{c} \begin{bmatrix} [T(\vec{v}_1)]_\beta & \cdots & [T(\vec{v}_n)]_\beta \end{bmatrix} \\ \begin{array}{ccc} \uparrow \text{col } 1 & \qquad\qquad & \uparrow \text{col } n \end{array} \end{array}$$

Ex. 1. Let $M_{2\times 2}(\mathbb{R})$ be the vector space of 2×2 matrices with real coefficients and let $\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ be its standard basis. Let $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by $T(A) = A^t$ where A^t is the transpose of the matrix A. Compute the matrix of T with respect to the basis α. $\to$ means $[T]_\alpha^\alpha$

only makes sense if $T: V \to V$

Recall: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ (the rows become the columns)

Let $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Then $[T]_\alpha^\alpha = \begin{bmatrix} [T(E_1)]_\alpha & [T(E_2)]_\alpha & [T(E_3)]_\alpha & [T(E_4)]_\alpha \end{bmatrix}$

$T(E_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^t = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = E_1 = 1E_1 + 0E_2 + 0E_3 + 0E_4 \Rightarrow [T(E_1)]_\alpha = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$T(E_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^t = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = E_3 \Rightarrow [T(E_2)]_\alpha = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$T(E_3) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^t = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = E_2 \Rightarrow [T(E_3)]_\alpha = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$T(E_4) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^t = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = E_4 \Rightarrow [T(E_4)]_\alpha = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

So $[T]_\alpha^\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Theorem: Let $T : V \to W$ be a linear transformation between finite-dimensional vector spaces V and W. If $A = [T]_\alpha^\beta$ where α and β are *any* bases of V and W, respectively, then

(1) T is one-to-one if and only if nullity$(A) = 0$.
(2) T is onto if and only if the rank$(A) = \dim(W)$.

Ex. 2. Let $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the transpose map from Ex. 1. Is T one-to one? Is T onto?

$$[T]_\alpha^\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R2 \leftrightarrow R3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since there are 4 pivots and no free variables, rank$(T) = 4$ and nullity $(T) = 0$. Since $\dim(M_{2\times 2}(\mathbb{R})) = 4$, T is onto. And since nullity$(T) = 0$, T is one-to-one as well.

Using the Matrix of T to find $T(\vec{v})$: Let $T : V \to W$ and α and β be bases of the vector spaces V and W, respectively. For a vector $\vec{v} \in V$, write down the equation that relates the coordinate vector of $T(\vec{v})$ to the coordinate vector of $\vec{v}$ and the matrix of T.

$$[T(\vec{v})]_\beta = [T]_\alpha^\beta [\vec{v}]_\alpha$$

Matrix of a Composition: Let $T : V \to W$ and $S : W \to X$ be linear transformations and let α, β, γ be bases of the vector spaces V, W, and X, respectively. Write down the equation that relates the matrix of the composition ST to the matrices of S and T.

$ST : V \to X$

$$[ST]_\alpha^\gamma = [S]_\beta^\gamma [T]_\alpha^\beta$$

output basis

input basis

out put

input

Ex. 3. Let $V = \mathbb{R}^4$ and $W = P_2(\mathbb{R})$ be the vector space of polynomials with real coefficients and degree at most 2. Let $\alpha = \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\}$ be the standard basis of $\mathbb{R}^4$ and let $\beta = \{1, x+1, x^2+x+1\}$. It is a fact, which you may assume, that β is a basis for W. Suppose that $T : V \to W$ is a linear transformation and the matrix of T with respect to α and β is $[T]_\alpha^\beta = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 0 & -1 \\ 0 & 2 & -3 & 1 \end{bmatrix}$. Find $T(0, 2, -1, 3)$.

Let $\vec{v} = (0,2,-1,3)$. We want $T(\vec{v})$ so we use $[T(\vec{v})]_\beta = [T]_\alpha^\beta [\vec{v}]_\alpha$.

$\vec{v} = 0\vec{e}_1 + 2\vec{e}_2 + -1\vec{e}_3 + 3\vec{e}_4 \Rightarrow [\vec{v}]_\alpha = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 3 \end{bmatrix}$.

Then
$$[T(\vec{v})]_\beta = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 0 & -1 \\ 0 & 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1\cdot 0 + 2\cdot 2 + 3(-1) + 4\cdot 3 \\ 2\cdot 0 + 0\cdot 2 + 0\cdot -1 + -1\cdot 3 \\ 0\cdot 0 + 2\cdot 2 + -3(-1) + 1\cdot 3 \end{bmatrix} = \begin{bmatrix} 13 \\ -3 \\ 10 \end{bmatrix}$$

These are the coefficients of $T(\vec{v})$ w.r.t. β so

$$T(\vec{v}) = 13(1) + -3(x+1) + 10(x^2+x+1)$$
$$= 13 - 3x - 3 + 10x^2 + 10x + 10$$
$$= 10x^2 + 7x + 20.$$

Invertible Matrices: Explain what it means for an $n \times n$ matrix A to be invertible. (Note: *Only* square matrices can be invertible.)

An $n \times n$ matrix A is invertible iff there is a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$ where $I = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$ is the $n \times n$ identity matrix.

Theorem: Let A be an $n \times n$ matrix. The following are equivalent:

(1) A is invertible.
(2) The columns of A are linearly independent.
(3) The rows of A span $\mathbb{R}^n$.
(4) The columnspace (i.e., range or image) of A is $\mathbb{R}^n$.
(5) The nullspace (i.e., kernel) of A is $\{\vec{0}\}$.
(6) $\text{rank}(A) = n$.
(7) $\text{nullity}(A) = 0$.
(8) $\det(A) \neq 0$.
(9) $\lambda = 0$ is not an eigenvalue of A.

Ex. 4. Consider the matrix $A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 2 & 4 \end{pmatrix}$.

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

(a) Compute $\det(A)$.
(b) Is A invertible? If so, compute the inverse of A.

a) $\det(A) = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 2 & 4 \end{vmatrix} = -0 \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} + 0 \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -1(1\cdot 2 - 1 \cdot 1) = -1$

expanding along row 2

b) Since $\det(A) \neq 0$, A is invertible. We compute:

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R1 - R3 \to R3} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & -2 & 1 & 0 & -1 \end{array}\right] \xrightarrow{R2 \leftrightarrow R3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R1 + R2 \to R1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & -1 & -2 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{2R3 + R2 \to R2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & -1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{-R2 \to R2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 & -2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

Then $A^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ -1 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Invertible Maps: Let $T : V \to W$ be a linear transformation. Explain what it means for T to be invertible.

A map $T: V \to W$ is invertible if there is a map $T^{-1}: W \to V$ such that $TT^{-1}(\vec{w}) = \vec{w}$ for all $\vec{w} \in W$ and $T^{-1}T(\vec{v}) = \vec{v}$ for all $\vec{v} \in V$.

(or $TT^{-1} = I_W$ and $T^{-1}T = I_V$)

Theorem: A map T is invertible if and only if T is one-to-one and onto.

Theorem: Let $T : V \to W$ be a linear transformation between finite-dimensional vector spaces. If $A = [T]_\alpha^\beta$ where α and β are *any* bases of V and W, respectively, then T is invertible if and only if A is invertible.

Ex. 5. Let $T : V \to V$ be a linear transformation and let $\alpha = \{\vec{v}_1, \vec{v}_2\}$ be a basis for the vector space V. Suppose that the matrix of T with respect to α is $[T]_\alpha^\alpha = \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}$.

(a) Explain how you know that T is invertible.
(b) Calculate $T^{-1}(\vec{v}_1)$. Write your answer as a linear combination of the vectors $\vec{v}_1$ and $\vec{v}_2$.

a) $\det([T]_\alpha^\alpha) = 3\cdot 1 - 2(-1) = 5 \neq 0$ so T is invertible.

b) $$\left[\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \xrightarrow{2R1-3R2 \to R2} \left[\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ 0 & -5 & 2 & -3 \end{array}\right] \xrightarrow{-5R1+R2 \to R1}$$

$$\left[\begin{array}{cc|cc} -15 & 0 & -3 & -3 \\ 0 & -5 & 2 & -3 \end{array}\right] \xrightarrow[-\frac{1}{5}R2 \to R2]{-\frac{1}{15}R1 \to R1} \left[\begin{array}{cc|cc} 1 & 0 & 1/5 & 1/5 \\ 0 & 1 & -2/5 & 3/5 \end{array}\right]$$

Then $[T^{-1}]_\alpha^\alpha = \begin{bmatrix} 1/5 & 1/5 \\ -2/5 & 3/5 \end{bmatrix} = \begin{bmatrix} [T^{-1}(\vec{v}_1)]_\alpha & [T^{-1}(\vec{v}_2)]_\alpha \end{bmatrix}$.

So $[T^{-1}(\vec{v}_1)]_\alpha = \begin{bmatrix} 1/5 \\ -2/5 \end{bmatrix}$. Then $T^{-1}(\vec{v}_1) = \frac{1}{5}\vec{v}_1 + \frac{-2}{5}\vec{v}_2$.

Alternatively, $[v_1]_\alpha = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so multiply $[T^{-1}]_\alpha^\alpha [v_1]_\alpha$ to get $[T^{-1}(\vec{v}_1)]_\alpha$.

Change of Basis: Recall that the identity map $I : V \to V$ satisfies $I(\vec{v}) = \vec{v}$ for all $\vec{v} \in V$. Let α and β both be bases of V. In this case, the matrix $[I]_\alpha^\beta$ is called the change of basis matrix from α to β.

Inverse of a Change of Basis Matrix: Since the identity map is invertible, so is any change of basis matrix. What is $\left([I]_\alpha^\beta\right)^{-1}$?

$$\left([I]_\alpha^\beta\right)^{-1} = [I]_\beta^\alpha.$$

Changing Coordinates for the Matrix of a Transformation: Suppose that $T : V \to W$, α and α' are bases for V and β and β' are bases for W. Given $[T]_\alpha^\beta$, write down expressions for $[T]_\alpha^{\beta'}$, $[T]_{\alpha'}^\beta$, and $[T]_{\alpha'}^{\beta'}$.

$$[T]_\alpha^{\beta'} = [I]_\beta^{\beta'}[T]_\alpha^\beta \qquad [T]_{\alpha'}^{\beta} = [T]_\alpha^\beta [I]_{\alpha'}^\alpha \qquad [T]_{\alpha'}^{\beta'} = [I]_\beta^{\beta'}[T]_\alpha^\beta [I]_{\alpha'}^\alpha$$

Ex. 6. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T\underbrace{\begin{pmatrix}1\\1\end{pmatrix}}_{v_1} = \begin{pmatrix}8\\-1\end{pmatrix}$ and $T\underbrace{\begin{pmatrix}1\\-1\end{pmatrix}}_{v_2} = \begin{pmatrix}-2\\3\end{pmatrix}$.

(a) Find the matrix of T with respect to the standard basis of $\mathbb{R}^2$.

(b) Is T one-to-one? Is T onto? Justify your answers.

Let $\alpha = \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}$. We have $T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}8\\-1\end{pmatrix} = \left[T\begin{pmatrix}1\\1\end{pmatrix}\right]_{std}$

and $T\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}-2\\3\end{pmatrix} = \left[T\begin{pmatrix}1\\-1\end{pmatrix}\right]_{std}$.

Then $[T]_\alpha^{std} = \begin{pmatrix}8 & -2\\-1 & 3\end{pmatrix}$,

and we want $[T]_{std}^{std} = [T]_\alpha^{std}[I]_{std}^\alpha$.

Now $[I]_\alpha^{std}$ is easier to find since $[I]_\alpha^{std} = \left[\,[I(v_1)]_{std} \;\; [I(v_2)]_{std}\,\right]$

$= \left[\,v_1 \;\; v_2\,\right]$

Then $[I]_\alpha^{std} = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$ so we compute the inverse.

$$\left[\begin{array}{cc|cc}1 & 1 & 1 & 0\\1 & -1 & 0 & 1\end{array}\right] \xrightarrow{R1-R2\to R2} \left[\begin{array}{cc|cc}1 & 1 & 1 & 0\\0 & 2 & 1 & -1\end{array}\right] \xrightarrow{2R1-R2\to R1} \left[\begin{array}{cc|cc}2 & 0 & 1 & 1\\0 & 2 & 1 & -1\end{array}\right]$$

So $[I]_{std}^\alpha = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$. Finally, $[T]_{std}^{std} = \begin{bmatrix}8 & -2\\-1 & 3\end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}6 & 10\\2 & -4\end{bmatrix}$

and $[T]_{std}^{std} = \begin{bmatrix}3 & 5\\1 & -2\end{bmatrix}$.

b) $\det\left([T]_{std}^{std}\right) = -6 - 5 = -11 \neq 0$ so T is invertible. Thus, T is both one-to-one and onto.

Additional Problems

Ex. 7. Let $P_n(\mathbb{R})$ be the vector space of polynomials with real coefficients of degree at most n. Define $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ by $T(f) = \int_0^x f(t)\, dt$.

(a) Compute the matrix of T with respect to the standard bases $\underset{\alpha_n}{\{1, x, x^2\}}$ of $P_2(\mathbb{R})$ and $\underset{\beta_n}{\{1, x, x^2, x^3\}}$ of $P_3(\mathbb{R})$.

(b) Is T one-to-one? yes Is T onto? no

(c) Find bases for $\ker(T)$ and $\mathrm{Im}(T)$. $\to \{x, x^2, x^3\}$

$$[T]_\alpha^\beta = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

Ex. 8. Let $M_{2\times 2}(\mathbb{R})$ be the vector space of 2×2 matrices with real coefficients and let

$\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ be its standard basis.

(a) Let $T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ by $T(A) = a + b + c + d$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Compute the matrix of T with respect to the bases α for $M_{2\times 2}(\mathbb{R})$ and $\beta = \{1\}$ for $\mathbb{R}$. $[T]_\alpha^\beta = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$

(b) Find a basis for $\ker(T)$. $\longrightarrow x_1 = -x_2 - x_3 - x_4$

(c) Is T one-to one? no Is T onto? yes

$$\Rightarrow x_2 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \Rightarrow \left\{ \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

Ex. 9. Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be the linear transformation with matrix representation $[T]_{std}^{std} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 5 & 5 & 7 & 7 & 7 \end{bmatrix}$ with respect to the standard bases on $\mathbb{R}^5$ and $\mathbb{R}^3$. Find a basis for $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$. see Ex. 4 from notes 2

Ex. 10. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (2x + y - 4z, 4y - 5z, -z)$. Determine whether or not T is invertible, and if so, find a formula for $T^{-1}(x, y, z)$. $= \left(\frac{1}{2}x - \frac{1}{8}y - \frac{11}{8}z, \frac{1}{4}y - \frac{5}{4}z, -z\right)$

Ex. 11. Let A and B be invertible $n \times n$ matrices. Show that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. (Note: This is a common result that you could usually use without proof.) check your HW §2.6 #5

Ex. 12. Let $\alpha = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and $\beta = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ be bases for a vector space V. Suppose that

$$\begin{aligned} \vec{v}_1 &= \vec{w}_1 - 2\vec{w}_2 - 2\vec{w}_3 \\ \vec{v}_2 &= -\vec{w}_2 - \vec{w}_3 \\ \vec{v}_3 &= 2\vec{w}_2 - \vec{w}_3. \end{aligned}$$

Compute $\begin{pmatrix} 1 & 0 & 0 \\ -2 & -1 & 2 \\ -2 & -1 & -1 \end{pmatrix}^{-1}$

Compute the change of basis matrix from β to α.

Ex. 13. Let P_2 be the vector space of polynomials with real coefficients of degree at most 2, and let $\alpha = \{1, x + 1, x^2 + x + 1\}$. It is a fact, which you may assume, that α is a basis for V. Suppose that $T : V \to V$ is a linear transformation and the matrix of T relative to α is $[T]_\alpha^\alpha = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & -1 \\ 2 & 5 & 1 \end{bmatrix}$. Find $T(3x^2 + x + 2)$. $= -5x^2 - 5x + 1$

Find $[T]_{std}^{std} = \begin{bmatrix} 6 & 1 & -4 \\ 5 & 0 & -5 \\ 2 & 3 & -4 \end{bmatrix}$ and mult. by $[p]_{std} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

Ex. 14. Let $P_2 = \{a + bt + ct^2 : a, b, c \in \mathbb{R}\}$ and $T : P_2 \to \mathbb{R}^2$ be defined by $T(p) = \begin{bmatrix} p(1) \\ p'(1) \end{bmatrix}$. You may assume that T is linear.

(a) Find the matrix representation of T with respect to the bases $\{1, t, t^2\}$ and $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$. $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$

(b) Find the $\overset{2}{\text{rank}}$ and $\overset{1}{\text{nullity}}$ of T.

(c) Find bases of the kernel and image of T. $= \mathbb{R}^2$ so take $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$$\begin{aligned} x_2 &= -2x_3 \\ x_1 &= -2x_2 - 3x_3 \\ &= -2(-2x_3) - 3x_3 \\ &= -7x_3 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ -2 \\ 1 \end{bmatrix} x_3 \Rightarrow (-7 + -2t + t^2)x_3$$

$$\Rightarrow \{-7 - 2t + t^2\}$$