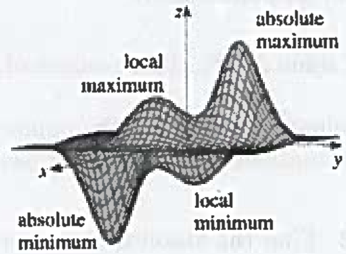
Maxima and Minima of Functions of Several Variables

(from Stewart, *Calculus*, Chapter 14)

Critical Points and Local Extrema



<u>Critical Point</u>: A point (a, b) in the domain of f where

$$f_x(a, b) = 0 \text{ and } f_y(a, b) = 0 \quad \text{i.e. } \nabla f(a,b) = \vec{0}$$

or where one of these partial derivatives does not exist.

Thm. If f has a local maximum or local minimum at (a, b) then (a, b) is a critical point of f.

Second Derivative Test. Suppose the second partial derivatives of f are continuous near (a, b) and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ (that is, (a, b) is a critical point of f). Let

$$D(x, y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = (f_{xx})(f_{yy}) - (f_{xy})^2$$

be the determinant of the <u>Hessian</u> matrix of second order partial derivatives of f.

$f_{xx} f_{yy} > 0$ same sign $\Leftarrow$

(a) If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum. $f_{xx} > 0 \Rightarrow$ concave up

(b) If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum. $f_{xx} < 0 \Rightarrow$ concave down

(c) If $D(a, b) < 0$, then (a, b) is a saddle point.

(d) If $D = 0$, the test gives no information - f could have a local maximum or local minimum at (a, b), or (a, b) could be a saddle point.

Ex. 1. Let $f(x, y) = x^4 + 2y^2 - 4xy$. Find all critical points of f, and classify each as a local maximum, local minimum, or saddle point.

$f_x(x,y) = 4x^3 - 4y$
$= 4x^3 - 4y = 0$
$\Rightarrow 4x^3 = 4y$
$x^3 = y$

$f_y(x,y) = 4y - 4x = 0$
$\Rightarrow y = x$

$f_{xx}(x,y) = 12x^2$
$f_{xy}(x,y) = -4$
$f_{yy}(x,y) = 4$

We have: $y = x$ and $y = x^3$ $\Rightarrow x = x^3$ or $x^3 - x = 0$
$x(x^2 - 1) = 0$
$x(x-1)(x+1) = 0$
$x = 0, 1, -1$

$x=0,\ y=0$ $D(0,0) = 0 - (-4)^2 = -16 < 0$
$f_{xx}(0,0) = 0$
So, f has a saddle point at $(0,0)$.

$x=1,\ y=1$ $D(1,1) = 12(4) - (-4)^2 > 0$
$f_{xx}(1,1) = 12 > 0$
So, f has a local min at $(1,1)$

$x=-1,\ y=-1$ $D(-1,-1) = 12(4) - 16 > 0$
$f_{xx}(-1,-1) = 12 > 0$
So, f has a local min at $(-1,-1)$.

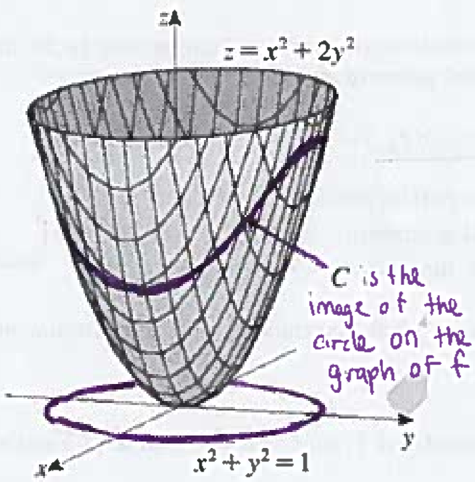
Method of Lagrange Multipliers

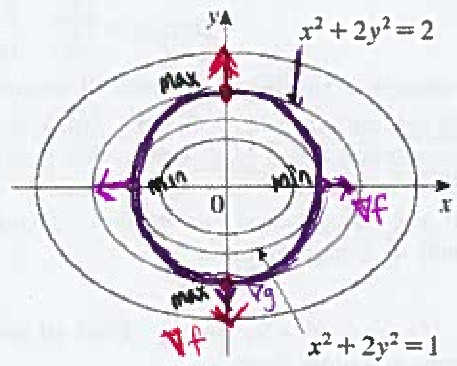
To find the maximum and minimum values of a function f subject to the constraint $g = k$ for some constant k (assuming these extreme values exist and $\nabla g \neq \vec{0}$ when $g = k$):

1. Find all points where
$$\nabla f = \lambda \nabla g \quad \text{and} \quad g = k$$
for some $\lambda \in \mathbb{R}$. These values of λ are called the Lagrange multipliers.

2. Evaluate f at each of the points from Step 1. The largest is the maximum value of f and the smallest is the minimum value of f, subject to the constraint.

Ex. 2. Find the absolute maximum and minimum values of $f(x, y) = x^2 + 2y^2$ on the circle $\underbrace{x^2 + y^2}_{g(x,y) = x^2+y^2} = 1$.





The level curves are tangent. Since the gradient is perpendicular to the level curves $\Rightarrow$ gradients are parallel.

$f(x,y) = x^2 + 2y^2$

$\nabla f(x,y) = \langle 2x, 4y \rangle$

$g(x,y) = x^2 + y^2$

$\nabla g(x,y) = \langle 2x, 2y \rangle$

$\nabla f = \lambda \nabla g \Rightarrow 2x = \lambda 2x$ and $4y = \lambda 2y$

$2x(\lambda - 1) = 0$	$2y(\lambda - 2) = 0$
$x = 0$ or $\lambda = 1$	$y = 0$ or $\lambda = 2$
$x^2+y^2=1 \Rightarrow y = \pm 1$; $4y = 2y \Rightarrow y = 0,\ x = \pm 1$	$x^2+y^2=1 \Rightarrow x = \pm 1$; $2x = 4x \Rightarrow x = 0,\ y = \pm 1$

notice this is redundant

All together, we have four points: $(0, \pm 1)$ and $(\pm 1, 0)$.

Evaluating, $f(0, \pm 1) = 2$ and $f(\pm 1, 0) = 1$.

Thus, the absolute max val. of f on the circle is 2, which occurs at the points $(0, \pm 1)$ and the abs. min. val. of f is 1, occuring at $(\pm 1, 0)$.

ex. $x^2+y^2 \le 1$
disk

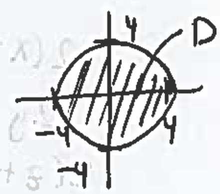
Absolute Maxima and Minima

Extreme Value Theorem for Functions of Two Variables. If $f(x, y)$ is continuous on a closed, bounded set, D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D. These extreme values occur either at the critical points of f or on the boundary of D.

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

(1) Find the values of f at each critical point in D.
(2) Find the maximum and minimum of $f(x, y)$ on the boundary of D.
(3) The largest of these values is the absolute maximum and the smallest is the absolute minimum.

Ex. 3. Find the points at which the absolute maximum and minimum values of the function $f(x, y) = 2x^2 + 3y^2 - 4x - 5$ on the disk $x^2 + y^2 \le 16$ occur. State all points where the extrema occur as well as the maximum and minimum values.



1) Critical points: $f_x(x,y) = 4x - 4 = 0 \Rightarrow x = 1$ $\qquad f_y(x,y) = 6y = 0 \Rightarrow y = 0$

One critical pt $(1,0)$ and $f(1,0) = 2 - 4 - 5 = -7$

2) Boundary: $x^2+y^2 = 16$ $\qquad$ Let $g(x,y) = x^2+y^2$

Lagrange Mult $\qquad \nabla f(x,y) = \langle 4x-4,\ 6y \rangle \qquad \nabla g(x,y) = \langle 2x, 2y \rangle$

$\nabla f = \lambda \nabla g \Rightarrow 4x - 4 = \lambda 2x$ and $6y = \lambda 2y$

$2y(\lambda - 3) = 0$

$y = 0$ or $\lambda = 3$

$y=0$	$\lambda = 3$
$x^2+y^2=16$	$4x-4 = 6x$
$\Rightarrow x = \pm 4$	$2x = -4$
	$x = -2$
	$4 + y^2 = 16$
	$y = \pm\sqrt{12}$

Evaluate: $f(4,0) = 32 - 16 - 5 = 11$

$f(-4,0) = 32 + 16 - 5 = 43$

$f(-2, \pm\sqrt{12}) = 8 + 36 + 8 - 5 = 47$

3) The absolute max value of f on D is 47, which occurs at $(-2, \pm\sqrt{12})$ and the abs. min val. of f is -7 at $(1,0)$.

Applications

Ex. 4. Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$.

Minimize $D = \sqrt{(x-1)^2 + (y-0)^2 + (z+2)^2}$ which is equivalent to minimizing

$f(x,y,z) = (x-1)^2 + y^2 + (z+2)^2$, subject to constraint $\underbrace{x + 2y + z}_{g(x,y,z)} = 4$.

$\nabla f = \lambda \nabla g \Rightarrow \langle 2(x-1), 2y, 2(z+2) \rangle = \lambda \langle 1, 2, 1 \rangle$

$2(x-1) = \lambda$

$2y = 2\lambda \Rightarrow y = \lambda = 2x - 2$

$2(z+2) = \lambda$ and $x - 1 = z + 2$

$\Rightarrow z = x - 3$

and $x + 2y + z = 4$

Then $x + 2y + z = x + 2(2x-2) + x - 3 = 4$

gives $6x - 7 = 4$

or $x = 11/6$

$y = \frac{11}{3} - 2 = 5/3$

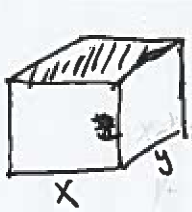
$z = \frac{11}{6} - 3 = -7/6$

So, the shortest distance is $D = \sqrt{\left(\frac{11}{6} - 1\right)^2 + \left(5/3\right)^2 + \left(-\frac{7}{6} + 2\right)^2}$

$D = \sqrt{(5/6)^2 + 25/9 + (5/6)^2}$

$= \sqrt{50/36 + 25/9} = \sqrt{\frac{25}{18} + \frac{25}{9}} = \sqrt{\frac{75}{18}} = \frac{5\sqrt{3}}{3\sqrt{2}} = \frac{5\sqrt{6}}{6}$

Ex. 5. A rectangular box without a lid is to be made from 12 m² of cardboard. Find the maximum volume of such a box.



Maximize $V(x,y,z) = xyz$ subject to $\overbrace{2xz + 2yz + xy}^{g(x,y,z)} = 12$.

$\nabla V = \lambda \nabla g \Rightarrow \langle yz, xz, xy \rangle = \lambda \langle 2z + y, 2z + x, 2x + 2y \rangle$

① $x \cdot yz = \lambda(2z + y) \cdot x$

② $y \cdot xz = \lambda(2z + x) \cdot y$

③ $z \cdot xy = \lambda(2x + 2y) \cdot z$

Note that $x, y, z \neq 0$ so we can multiply by each & exploit the symmetry

$\Rightarrow 2\lambda xz + \cancel{xy} = 2\lambda yz + xy$

$2\lambda xz = 2\lambda yz$

$2\lambda x = 2\lambda y$ since $z \neq 0$

$\lambda(x - y) = 0$

$\lambda = 0$ or $x = y$

Impossible b.c. then x, y, or z must be 0

① & ③ $\Rightarrow$

$2\cancel{\lambda}\cancel{x}z + xy = 2\cancel{\lambda}\cancel{x}z + 2yz$

$xy = 2yz$

$x = 2z$

Then $x = y = 2z$ so

$2(2z)z + 2(2z)z + 2z(2z) = 12$

$12z^2 = 12$

$z = 1 \Rightarrow z = \pm 1$

$z = -1$ is not possible so $z = 1$.

$x = y = 2$.

Then the max. volume is $V(2,2,1) = 4 \text{ m}^3$.