3. Matrices

(References: Comps Study Guide for Linear Algebra Section 3;
Damiano & Little, *A Course in Linear Algebra*, Chapters 2 and 3)

Coordinate Vectors: Let V be a vector space and $\alpha = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for V. Then for any $\vec{v} \in V$, there are unique coefficients $a_1, \ldots, a_n \in \mathbb{R}$ such that

$$\vec{v} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n.$$

In this notation, the coordinate vector of $\vec{v}$ with respect to α is $[\vec{v}]_\alpha =$

Matrix of a Linear Transformation: Let $T : V \to W$ be a linear transformation and let $\alpha = \{\vec{v}_1, \ldots, \vec{v}_n\}$ and $\beta = \{\vec{w}_1, \ldots, \vec{w}_n\}$ be bases for V and W, respectively. Then the matrix of T with respect to α and β, denoted $[T]_\alpha^\beta$, is the matrix whose ith column is $[T(\vec{v}_i)]_\beta$, the coordinate vector of $T(\vec{v}_i)$ with respect to β.

Ex. 1. Let $M_{2\times 2}(\mathbb{R})$ be the vector space of 2×2 matrices with real coefficients and let $\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ be its standard basis. Let $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by $T(A) = A^t$ where A^t is the transpose of the matrix A. Compute the matrix of T with respect to the basis α.

Theorem: Let $T : V \to W$ be a linear transformation between finite-dimensional vector spaces V and W. If $A = [T]_\alpha^\beta$ where α and β are *any* bases of V and W, respectively, then

(1) T is one-to-one if and only if nullity$(A) = 0$.

(2) T is onto if and only if the rank$(A) = \dim(W)$.

Ex. 2. Let $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the transpose map from Ex. 1. Is T one-to one? Is T onto?

Using the Matrix of T to find $T(\vec{v})$: Let $T : V \to W$ and α and β be bases of the vector spaces V and W, respectively. For a vector $\vec{v} \in V$, write down the equation that relates the coordinate vector of $T(\vec{v})$ to the coordinate vector of $\vec{v}$ and the matrix of T.

Matrix of a Composition: Let $T : V \to W$ and $S : W \to X$ be linear transformations and let α, β, γ be bases of the vector spaces V, W, and X, respectively. Write down the equation that relates the matrix of the composition ST to the matrices of S and T.

Ex. 3. Let $V = \mathbb{R}^4$ and $W = P_2(\mathbb{R})$ be the vector space of polynomials with real coefficients and degree at most 2. Let $\alpha = \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\}$ be the standard basis of $\mathbb{R}^4$ and let $\beta = \{1, x+1, x^2+x+1\}$. It is a fact, which you may assume, that β is a basis for W. Suppose that $T : V \to W$ is a linear transformation and the matrix of T with respect to α and β is $[T]_\alpha^\beta = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 0 & -1 \\ 0 & 2 & -3 & 1 \end{bmatrix}$. Find $T(0, 2, -1, 3)$.

Invertible Matrices: Explain what it means for an $n \times n$ matrix A to be invertible. (Note: *Only* square matrices can be invertible.)

Theorem: Let A be an $n \times n$ matrix. The following are equivalent:

(1) A is invertible.
(2) The columns of A are linearly independent.
(3) The rows of A span $\mathbb{R}^n$.
(4) The columnspace (i.e., range or image) of A is $\mathbb{R}^n$.
(5) The nullspace (i.e., kernel) of A is $\{0\}$.
(6) $\text{rank}(A) = n$.
(7) $\text{nullity}(A) = 0$.
(8) $\det(A) \neq 0$.
(9) $\lambda = 0$ is not an eigenvalue of A.

Ex. 4. Consider the matrix $A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 2 & 4 \end{pmatrix}$.

(a) Compute $\det(A)$.
(b) Is A invertible? If so, compute the inverse of A.

Invertible Maps: Let $T : V \to W$ be a linear transformation. Explain what it means for T to be invertible.

Theorem: A map T is invertible if and only if T is one-to-one and onto.

Theorem: Let $T : V \to W$ be a linear transformation between finite-dimensional vector spaces. If $A = [T]_\alpha^\beta$ where α and β are *any* bases of V and W, respectively, then T is invertible if and only if A is invertible.

Ex. 5. Let $T : V \to V$ be a linear transformation and let $\alpha = \{\vec{v}_1, \vec{v}_2\}$ be a basis for the vector space V. Suppose that the matrix of T with respect to α is $[T]_\alpha^\alpha = \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}$.

(a) Explain how you know that T is invertible.
(b) Calculate $T^{-1}(\vec{v}_1)$. Write your answer as a linear combination of the vectors $\vec{v}_1$ and $\vec{v}_2$.

Change of Basis: Recall that the identity map $I : V \to V$ satisfies $I(\vec{v}) = \vec{v}$ for all $\vec{v} \in V$. Let α and β both be bases of V. In this case, the matrix $[I]_\alpha^\beta$ is called the change of basis matrix from α to β.

Inverse of a Change of Basis Matrix: Since the identity map is invertible, so is any change of basis matrix. What is $\left([I]_\alpha^\beta\right)^{-1}$?

Changing Coordinates for the Matrix of a Transformation: Suppose that $T : V \to W$, α and α' are bases for V and β and β' are bases for W. Given $[T]_\alpha^\beta$, write down expressions for $[T]_\alpha^{\beta'}$, $[T]_{\alpha'}^\beta$, and $[T]_{\alpha'}^{\beta'}$.

Ex. 6. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \end{pmatrix}$ and $T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

(a) Find the matrix of T with respect to the standard basis of $\mathbb{R}^2$.
(b) Is T one-to-one? Is T onto? Justify your answers.

Additional Problems

Ex. 7. Let $P_n(\mathbb{R})$ be the vector space of polynomials with real coefficients of degree at most n. Define $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ by $T(f) = \int_0^x f(t)\, dt$.

(a) Compute the matrix of T with respect to the standard bases $\{1, x, x^2\}$ of $P_2(\mathbb{R})$ and $\{1, x, x^2, x^3\}$ of $P_3(\mathbb{R})$.
(b) Is T one-to one? Is T onto?
(c) Find bases for $\ker(T)$ and $\operatorname{Im}(T)$.

Ex. 8. Let $M_{2\times 2}(\mathbb{R})$ be the vector space of 2×2 matrices with real coefficients and let $\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ be its standard basis.

(a) Let $T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ by $T(A) = a + b + c + d$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Compute the matrix of T with respect to the bases α for $M_{2\times 2}(\mathbb{R})$ and $\beta = \{1\}$ for $\mathbb{R}$.
(b) Find a basis for $\ker(T)$.
(c) Is T one-to one? Is T onto?

Ex. 9. Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be the linear transformation with matrix representation $[T]_{std}^{std} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 5 & 5 & 7 & 7 & 7 \end{bmatrix}$ with respect to the standard bases on $\mathbb{R}^5$ and $\mathbb{R}^3$. Find a basis for $\operatorname{Ker}(T)$ and $\operatorname{Im}(T)$.

Ex. 10. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (2x + y - 4z, 4y - 5z, -z)$. Determine whether or not T is invertible, and if so, find a formula for $T^{-1}(x, y, z)$.

Ex. 11. Let A and B be invertible $n \times n$ matrices. Show that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. (Note: This is a common result that you could usually use without proof.)

Ex. 12. Let $\alpha = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and $\beta = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ be bases for a vector space V. Suppose that

$$\begin{aligned} \vec{v}_1 &= \vec{w}_1 - 2\vec{w}_2 - 2\vec{w}_3 \\ \vec{v}_2 &= -\vec{w}_2 - \vec{w}_3 \\ \vec{v}_3 &= 2\vec{w}_2 - \vec{w}_3. \end{aligned}$$

Compute the change of basis matrix from β to α.

Ex. 13. Let P_2 be the vector space of polynomials with real coefficients of degree at most 2, and let $\alpha = \{1, x + 1, x^2 + x + 1\}$. It is a fact, which you may assume, that α is a basis for V. Suppose that $T : V \to V$ is a linear transformation and the matrix of T relative to α is $[T]_\alpha^\alpha = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & -1 \\ 2 & 5 & 1 \end{bmatrix}$. Find $T(3x^2 + x + 2)$.

Ex. 14. Let $P_2 = \{a + bt + ct^2 : a, b, c \in \mathbb{R}\}$ and $T : P_2 \to \mathbb{R}^2$ be defined by $T(p) = \begin{bmatrix} p(1) \\ p'(1) \end{bmatrix}$. You may assume that T is linear.

(a) Find the matrix representation of T with respect to the bases $\{1, t, t^2\}$ and $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.
(b) Find the rank and nullity of T.
(c) Find bases of the kernel and image of T.