3. Maxima and Minima of Functions of Several Variables

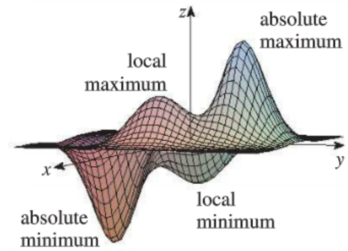
(from Stewart, *Calculus*, Chapter 14)

Critical Points and Local Extrema

Critical Point: A point (a, b) in the domain of f where

$$f_x(a, b) = 0 \text{ and } f_y(a, b) = 0$$

or where one of these partial derivatives does not exist.



Thm. If f has a local maximum or local minimum at (a, b) then (a, b) is a critical point of f.

Second Derivative Test. Suppose the second partial derivatives of f are continuous near (a, b) and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ (that is, (a, b) is a critical point of f). Let

$$D(x, y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = (f_{xx})(f_{yy}) - (f_{xy})^2$$

be the determinant of the Hessian matrix of second order partial derivatives of f.

(a) If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
(b) If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
(c) If $D(a, b) < 0$, then (a, b) is a saddle point.
(d) If $D = 0$, the test gives no information - f could have a local maximum or local minimum at (a, b), or (a, b) could be a saddle point.

Ex. 1. Let $f(x, y) = x^4 + 2y^2 - 4xy$. Find all critical points of f, and classify each as a local maximum, local minimum, or saddle point.

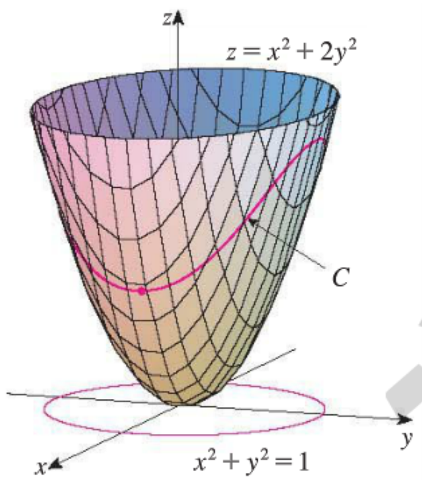
Method of Lagrange Multipliers

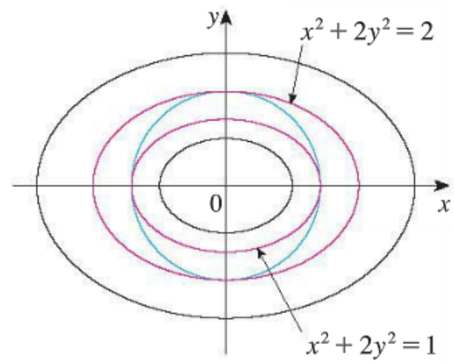
To find the maximum and minimum values of a function f subject to the constraint $g = k$ for some constant k (assuming these extreme values exist and $\nabla g \neq \vec{0}$ when $g = k$):

1. Find all points where
$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g = k$$
for some $\lambda \in \mathbb{R}$. These values of λ are called the Lagrange multipliers.

2. Evaluate f at each of the points from Step 1. The largest is the maximum value of f and the smallest is the minimum value of f, subject to the constraint.

Ex. 2. Find the absolute maximum and minimum values of $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.





Absolute Maxima and Minima

Extreme Value Theorem for Functions of Two Variables. If $f(x, y)$ is continuous on a closed, bounded set, D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D. These extreme values occur either at the critical points of f or on the boundary of D.

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

(1) Find the values of f at each critical point in D.
(2) Find the maximum and minimum of $f(x, y)$ on the boundary of D.
(3) The largest of these values is the absolute maximum and the smallest is the absolute minimum.

Ex. 3. Find the points at which the absolute maximum and minimum values of the function $f(x, y) = 2x^2 + 3y^2 - 4x - 5$ on the disk $x^2 + y^2 \leq 16$ occur. State all points where the extrema occur as well as the maximum and minimum values.

Applications

Ex. 4. Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$.

Ex. 5. A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.