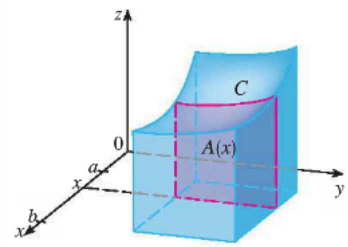
4. Double Integrals

(from Stewart, *Calculus*, Chapter 15)

Partial Integrals: $\int_c^d f(x,y)\ dy$ is calculated by holding x constant and integrating with respect to y from $y = c$ to $y = d$.

Note that the result is a function of x.

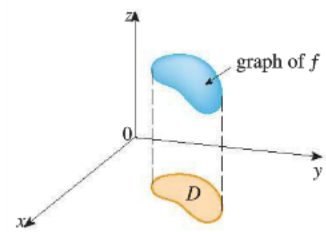
For each fixed x, the trace of $f(x, y)$ is a curve C in the plane x = constant. The partial integral $A(x) = \int_c^d f(x,y)\ dy$ is the area under the curve C from $y = c$ to $y = d$.



Similarly, $\int_a^b f(x,y)\ dx$ is calculated by holding y constant and integrating with respect to x from $x = a$ to $x = b$. The result is a function of y.

Finding Volume with a Double Integral:

$\iint\limits_D f(x,y)dA$ is the signed volume between the surface $z = f(x, y)$ and the region D in the xy-plane.



Finding Area with a Double Integral:

$\text{Area}(D) = \iint\limits_D 1\ dA$

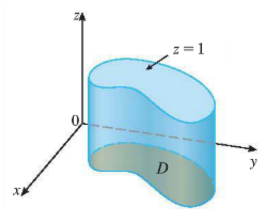


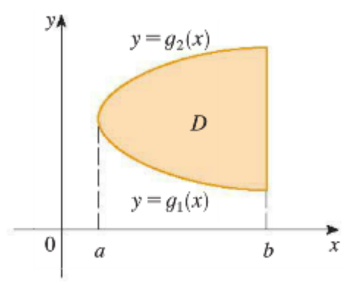
FIGURE 19
Cylinder with base *D* and height 1

Iterated Integrals

Vertically Simple Regions:

$D = \{(x,y)\ :\ a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

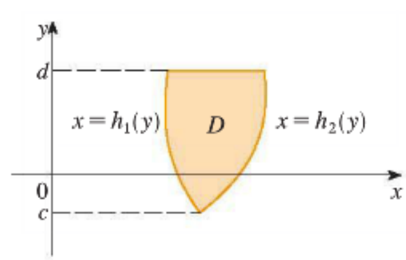
$$\iint_D f(x,y)\ dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ dy\ dx$$



Horizontally Simple Regions:

$D = \{(x,y)\ :\ c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$

$$\iint_D f(x,y)\ dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\ dx\ dy$$



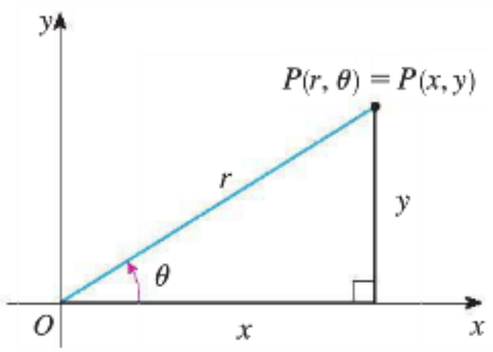
Ex. 1. Evaluate $\iint_D xy\ dA$ where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Ex. 2. Evaluate $\int_0^1 \int_x^1 \sin(y^2)\ dy\ dx$.

Ex. 3. Reverse the order of integration in the integral $\int_4^9 \int_2^{\sqrt{y}} f(x,y)\ dx\ dy$.

Ex. 4. Evaluate $\iint_D (x+2y)\ dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Polar Coordinates (r, θ)



r is the signed distance from the origin

θ is the angle measured counter-clockwise from the positive x-axis

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

$$r^2 = x^2 + y^2$$

$$\tan(\theta) = \frac{y}{x}$$

Double Integrals in Polar Coordinates:

$$\iint_R f(x,y)\ dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\ r\ dr\ d\theta.$$

Ex. 5. Evaluate $\displaystyle\iint_R (3x + 4y^2)dA$ where R is the region in the upper half plane $y \geq 0$ bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Ex. 6. Evaluate the integral $\int_{-1}^{0}\int_{0}^{\sqrt{1-x^2}} \cos(x^2+y^2)\, dy\, dx$.

Ex. 7. Find the volume of the region bounded by the paraboloids $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$.

Ex. 8. Evaluate the integrals.

(a) $\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\, dy\, dx$

(b) $\iint_D e^{x^2}\, dA$ where D is the region bounded by the lines $y = 2x$, $y = 0$, and $x = 1$.

(c) $\iint_D e^{x^2+y^2}\, dA$ where D is the top half of the disk centered at the origin of radius 3.