Additional Problems

Ex. 7. Consider the matrix $A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 0 & -5 \\ 1 & 0 & 0 \end{bmatrix}$.

(a) Find all eigenvalues of A.

(b) Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$, or show that no such matrices exist.

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

a) $$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 0 & -1 \\ 3 & -\lambda & -5 \\ 1 & 0 & -\lambda \end{vmatrix} = -0 + -\lambda \begin{vmatrix} 2-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} - 0$$

$$= -\lambda\left[(2-\lambda)(-\lambda)+1\right]$$

$$= -\lambda\left(\lambda^2 - 2\lambda + 1\right)$$

$$= -\lambda(\lambda-1)^2 = 0$$

$$\Rightarrow \lambda = 0, 1$$

b) $\underline{\lambda = 1}$ $\quad (A - 1I)\vec{v} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 3 & -1 & -5 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \xrightarrow[\text{new } R2 - R3 \to R2]{R1 \leftrightarrow R2} \left[\begin{array}{ccc|c} 3 & -1 & -5 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R1 - 3R2 \to R2$$

$$\left[\begin{array}{ccc|c} 3 & -1 & -5 & 0 \\ 0 & -3 & -14 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since there are pivots in columns 1 & 2, there is only one free variable and hence $\dim(E_1) = 1$. Thus, since the algebraic multiplicity of $\lambda = 1$ is 2, we know that A is not diagonalizable. So, there are no such matrices D and P.

matrices exist.

Ex. 8. Let A be the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ -6 & -1 & -3 \end{bmatrix}$.

(a) Find all eigenvalues of A.
(b) Find a a basis of each eigenspace.
(c) Find a diagonal matrix D and an invertible matrix P such that $D = PAP^{-1}$, or show that no such matrices exist.

c) $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $\quad P = \begin{pmatrix} -1/2 & 2/3 & -1/3 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

a) $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 2-\lambda & 0 \\ -6 & -1 & -3-\lambda \end{vmatrix} = -0 + (2-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ -6 & -3-\lambda \end{vmatrix} - 0$

$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

$= (2-\lambda)\left[(2-\lambda)(-3-\lambda) + 6\right]$

$= (2-\lambda)\left[\lambda^2 + \lambda - 6 + 6\right]$

$= (2-\lambda)(\lambda)(\lambda+1) = 0$

$\Rightarrow \quad \lambda = 0, 2, -1$

b) $\underline{\lambda = 0}$ $\quad (A - 0I)\vec{v} = \vec{0}$

$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ -6 & -1 & -3 & 0 \end{array}\right]$ $\xrightarrow{3R1 + R3 \to R3}$ $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Then a basis for E_0 is $\left\{\begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix}\right\}$.

So $\quad 2v_2 = 0 \Rightarrow v_2 = 0$

$2v_1 + v_2 + v_3 = 0 \Rightarrow v_1 = -\frac{1}{2}v_3$, v_3 free

$\underline{\lambda = 2}$ $\quad (A - 2I)\vec{v} = \vec{0}$

$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -6 & -1 & -5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -6 & -1 & -5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$ $-6v_1 - v_2 - 5v_3 = 0$

$v_2 = -v_3$

$\Rightarrow -6v_1 = 4v_3 \Rightarrow v_1 = \frac{2}{3}v_3$

v_3 free

So a basis for E_2 is $\left\{\begin{pmatrix} 2/3 \\ -1 \\ 1 \end{pmatrix}\right\}$.

$\underline{\lambda = -1}$ $\quad (A + I)\vec{v} = \vec{0}$

$\left[\begin{array}{ccc|c} 3 & 1 & 1 & 0 \\ 0 & 3 & 0 & 0 \\ -6 & -1 & -2 & 0 \end{array}\right]$ $\xrightarrow{2R1 + R3 \to R3}$ $\left[\begin{array}{ccc|c} 3 & 1 & 1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

So a basis for E_{-1} is $\left\{\begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix}\right\}$.

$\Rightarrow \quad 3v_1 + v_2 + v_3 = 0 \quad \Rightarrow \quad v_1 = -\frac{1}{3}v_3$

$v_2 = 0 \qquad\qquad\qquad\qquad v_2 = 0$

v_3 free

Ex. 9. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix}.$$

Find a basis for $\mathbb{R}^3$ consisting of eigenvectors of A, or else prove that there is no such basis.

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 0 & -2 \\ 0 & -\lambda & 0 \\ -2 & 0 & 4-\lambda \end{vmatrix} = -0 + -\lambda \begin{vmatrix} 1-\lambda & -2 \\ -2 & 4-\lambda \end{vmatrix} - 0$$

$$= -\lambda\left[(1-\lambda)(4-\lambda) - 4\right]$$

$$= -\lambda\left[\lambda^2 - 5\lambda + 4 - 4\right]$$

$$= -\lambda^2(\lambda - 5) \implies \lambda = 0, 5$$

$\underline{\lambda = 0}$ $\quad (A - 0I)\vec{v} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 4 & 0 \end{array}\right] \xrightarrow{2R1 + R3 \to R3} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \begin{array}{l} v_1 = 2v_3 \\ v_2, v_3 \text{ free} \end{array}$$

Since $\dim(E_0) = 2$, we know there is an eigenbasis, so we continue finding the eigenvectors. A basis of E_0 is $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\}$.

$\underline{\lambda = 5}$ $\quad (A - 5I)\vec{v} = \vec{0}$

$$\left[\begin{array}{ccc|c} -4 & 0 & -2 & 0 \\ 0 & -5 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{array}\right] \xrightarrow{R1 - 2R3 \to R3} \left[\begin{array}{ccc|c} -4 & 0 & -2 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \begin{array}{l} -4v_1 = 2v_3 \implies v_1 = \frac{-1}{2}v_3 \\ v_2 = 0 \\ v_3 \text{ free} \end{array}$$

So a basis of E_5 is $\left\{ \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Thus, $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^3$ consisting of eigenvectors of A.

Ex. 10. Consider the matrix

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 5 & 0 \\ 1 & 2 & 2 \end{bmatrix}.$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Is A diagonalizable? Why or why not?

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -2 & 0 \\ 2 & 5-\lambda & 0 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0 - 0 + (2-\lambda)\begin{vmatrix} 1-\lambda & -2 \\ 2 & 5-\lambda \end{vmatrix}$$

$$= (2-\lambda)\left[(1-\lambda)(5-\lambda) + 4\right]$$

$$= (2-\lambda)\left[\lambda^2 - 6\lambda + 5 + 4\right]$$

$$= (2-\lambda)(\lambda^2 - 6\lambda + 9)$$

$$= (2-\lambda)(\lambda-3)^2$$

$\underline{\lambda = 3}$ $(A-3I)\vec{v} = \vec{0}$

$$\left[\begin{array}{ccc|c} -2 & -2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 1 & 2 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{2R1-R2 \to R2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since there are two pivot columns, there is only one free variable and so $\dim(E_3) = 1$. But the geometric multiplicity of $\lambda = 3$ is 2. Hence, A is not diagonalizable.

Ex. 11. Let V be a finite-dimensional vector space, and let $T : V \to V$ be a linear transformation. Prove that 0 is an eigenvalue of T if and only if the image (i.e., range) of T is *not* equal to V.

Let V be a finite-dimensional vector space with $\dim(V) = n$ and $T: V \to V$ be linear.

First suppose that $\lambda = 0$ is an eigenvalue of T. Then there is a nonzero vector $\vec{v} \in V \setminus \{\vec{0}\}$ such that $T(\vec{v}) = 0\vec{v} = \vec{0}$. Thus, $\vec{v} \in \mathrm{Ker}(T)$ and $\vec{v} \neq \vec{0}$, so the nullity of T is at least 1. Then by the rank-nullity theorem, the rank of T can be at most $n-1$. Hence, the $\mathrm{rank}(T)$ cannot be equal to $\dim(V)$, and consequently, the Image of T is not equal to V.

Now suppose that $\mathrm{Im}(T) \neq V$. Since $\mathrm{Im}(T)$ is a subspace of V, the $\dim(\mathrm{Im}(T))$ must then be strictly less than $n = \dim(V)$. So, $\mathrm{rank}(T)$ is at most $n-1$ and by the rank-nullity theorem, the nullity of T is at least 1. So, $\mathrm{Ker}(T) \neq \{\vec{0}\}$, i.e. there is a nonzero vector $\vec{v} \in V$ with $T(\vec{v}) = \vec{0}$. Then we have $T(\vec{v}) = 0 \cdot \vec{v}$ and $\vec{v} \neq \vec{0}$, so $\vec{v}$ is an eigenvector of T with eigenvalue $\lambda = 0$. Thus, 0 is an eigenvalue of T.

Ex. 12. Suppose that a 3×3 matrix A has 0 as an eigenvalue.

(a) What are the possible values of the rank of A? Justify your answer.

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by $T(x) = Ax$. Can T possibly be one-to-one? Can T be onto? Justify your answer.

a) Since 0 is an eigenvalue of A, there is a nonzero vector $\vec{v}$ such that $A\vec{v} = 0\vec{v} = \vec{0}$. Thus, $\mathrm{nullity}(A) \geq 1$. By the rank-nullity theorem, $\mathrm{rank}(A) = 3 - \mathrm{nullity}(A)$ and hence $\mathrm{rank}(A) \leq 2$. So, $\mathrm{rank}(A) = 0, 1,$ or 2.

b) By part A, the $\mathrm{nullity}(A) = \mathrm{nullity}(T) \geq 1$, so T cannot be one-to-one. Then since $T: \mathbb{R}^3 \to \mathbb{R}^3$, T cannot be onto either.

Ex. 13. Let A, B be $n \times n$ matrices that commute, i.e. $AB = BA$. Let $v \in \mathbb{R}^n$ be an eigenvector of A such that $Bv \neq 0$. Prove that Bv is also an eigenvector of A.

Let $A, B \in M_{n\times n}(\mathbb{R})$ and $AB = BA$. Suppose that $\vec{v} \in \mathbb{R}^n$ is an eigenvector of A such that $B\vec{v} \neq \vec{0}$. Then $\vec{v} \neq \vec{0}$ and there is a $\lambda \in \mathbb{R}$ such that $A\vec{v} = \lambda\vec{v}$. Multiplying by B, we have $B(A\vec{v}) = B(\lambda\vec{v})$ or $(BA)\vec{v} = \lambda(B\vec{v})$. Since $BA = AB$, $(AB)\vec{v} = \lambda(B\vec{v})$ and so $A(B\vec{v}) = \lambda(B\vec{v})$.
Then since $B\vec{v} \neq \vec{0}$, $B\vec{v}$ is an eigenvector of A with eigenvalue λ.

Ex. 14. Prove that if $T : V \to V$ is a linear map then the eigenspace E_0 corresponding to the eigenvalue $\lambda = 0$ is equal to $\mathrm{Ker}(T)$.

Let $T : V \to V$ be linear and E_0 be the eigenspace corresponding to $\lambda = 0$.
Let $\vec{v} \in E_0$. Then $T(\vec{v}) = 0\vec{v} = \vec{0}$ and so $\vec{v} \in \ker(T)$. Thus, $E_0 \subseteq \ker(T)$.
Now let $\vec{v} \in \ker(T)$. Then $T(\vec{v}) = \vec{0} = 0\vec{v}$. Hence, $\vec{v} \in E_0$ and so $\ker(T) \subseteq E_0$.
Thus, $E_0 = \ker(T)$.

Ex. 15. Suppose that A is an $n \times n$ matrix that satisfies $A^2 = I$, where I is the $n \times n$ identity matrix. Show that if λ is an eigenvalue of A then $\lambda = 1$ or $\lambda = -1$.

Suppose that λ is an eigenvalue of A where $A \in M_{n\times n}(\mathbb{R})$ and $A^2 = I$. Then there exists a nonzero vector $\vec{v} \in \mathbb{R}^n \setminus \{\vec{0}\}$ such that $A\vec{v} = \lambda\vec{v}$. Multiplying by A, $A^2\vec{v} = A(\lambda\vec{v})$ or $I\vec{v} = \lambda(A\vec{v})$ since $A^2 = I$. So, $\vec{v} = \lambda(\lambda\vec{v}) = \lambda^2\vec{v}$ since $A\vec{v} = \lambda\vec{v}$. Subtracting, we have $(\lambda^2 - 1)\vec{v} = \vec{0}$. Since $\vec{v} \neq \vec{0}$, $\lambda^2 - 1 = 0$ and hence $\lambda = 1$ or $\lambda = -1$.

Ex. 16. Let A be an $n \times n$ matrix and λ be an eigenvalue of A. Prove that λ^m is an eigenvalue of A^m for all integers $m \geq 1$.

Let $A \in M_{n\times n}(\mathbb{R})$ and λ be an eigenvalue of A. Then there is a nonzero vector $\vec{v} \in \mathbb{R}^n \setminus \{\vec{0}\}$ such that $A\vec{v} = \lambda\vec{v}$. We will use induction to show that $A^m\vec{v} = \lambda^m\vec{v}$ for all integers $m \geq 1$.

Base case: For $m=1$, we have $A^1\vec{v} = \lambda^1\vec{v}$ by definition of $\vec{v}$.

Inductive step: Suppose that $A^m\vec{v} = \lambda^m\vec{v}$ for some integer $m \geq 1$.

Multiplying by A, $A^{m+1}\vec{v} = A(\lambda^m\vec{v}) = \lambda^m(A\vec{v}) = \lambda^m(\lambda\vec{v}) = \lambda^{m+1}\vec{v}$.

Hence $A^{m+1}\vec{v} = \lambda^{m+1}\vec{v}$. Thus, by induction, $A^m\vec{v} = \lambda^m\vec{v}$ for all integers $m \geq 1$. Since $\vec{v} \neq \vec{0}$, it follows that λ^m is an eigenvalue of A^m for all integers $m \geq 1$.

Ex. 17. Let A be an $n \times n$ matrix and $\alpha \in \mathbb{R}$ be a scalar that is NOT an eigenvalue of A. Suppose that μ is an eigenvalue for the matrix $B = (A - \alpha I)^{-1}$ with corresponding eigenvector v. Prove that v is also an eigenvector for A and find a formula for the corresponding eigenvalue of A in terms of μ and α.

Let $A \in M_{n\times n}(\mathbb{R})$ and $\alpha \in \mathbb{R}$ be a scalar that is not an eigenvalue of A.* Suppose that μ is an eigenvalue for $B = (A-\alpha I)^{-1}$ with corresponding eigenvector $\vec{v}$. Then $\vec{v} \neq \vec{0}$ and $B\vec{v} = (A-\alpha I)^{-1}\vec{v} = \mu\vec{v}$.

(* Note that this means $\det(A-\alpha I) \neq 0$ and hence $A-\alpha I$ is invertible.)

Multiplying by $A-\alpha I$ gives, $\vec{v} = (A-\alpha I)\mu\vec{v} = \mu A\vec{v} - \alpha\mu\vec{v}$. Then $\mu A\vec{v} = (1+\alpha\mu)\vec{v}$.

Now since $B = (A-\alpha I)^{-1}$, B is invertible and since μ is an eigenvalue of B, $\mu \neq 0$. Thus, we can divide by μ to get $A\vec{v} = \frac{1+\alpha\mu}{\mu}\vec{v}$. Thus, since $\vec{v} \neq \vec{0}$, $\vec{v}$ is an eigenvector for A with eigenvalue $\lambda = \frac{1+\alpha\mu}{\mu}$.