4. Eigenvalues and Eigenvectors

(References: Comps Study Guide for Linear Algebra Section 4;
Damiano & Little, *A Course in Linear Algebra*, Chapter 4)

Let A be an $n \times n$ matrix, $\lambda \in \mathbb{R}$ be a scalar, and let $\vec{v} \in \mathbb{R}$. To say $\vec{v}$ is an eigenvector of A with eigenvalue λ means $\vec{v} \neq 0$ and $A\vec{v} = \lambda\vec{v}$.

Eigenvalues: Write down what it means to say that λ is an eigenvalue of A.

It means there is a nonzero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$.

Eigenvector: Write down what it means to say that $\vec{v}$ is an eigenvector of A.

It means that $\vec{v} \neq \vec{0}$ and there is a scalar $\lambda \in \mathbb{R}$ such that $A\vec{v} = \lambda\vec{v}$.

Ex. 1. Let A be an $n \times n$ matrix. Prove that A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A. (Note: This is a common theorem that you could usually use without proof.)

By Contrapositive

$\Rightarrow$ Suppose that $\lambda = 0$ is an eigenvalue of A. Then there is a nonzero vector $\vec{v}$ such that $A\vec{v} = 0\vec{v} = \vec{0}$. Then $\vec{v} \in \ker(A)$ and hence nullity$(A) \neq 0$ since $\vec{v} \neq \vec{0}$. Thus, A is not invertible. Taking the contrapositive, we have proven That if A is invertible then $\lambda = 0$ is not an eigenvalue of A.

$\Leftarrow$ Now suppose that $\lambda = 0$ is not an eigenvalue of A. Then there is no nonzero vector $\vec{v}$ such that $A\vec{v} = 0\vec{v} = \vec{0}$. Hence, $\ker(A) = \{\vec{0}\}$ and So nullity$(A) = 0$. Thus, A is invertible.

Ex. 2. Suppose that A is an invertible $n \times n$ matrix. Prove that if $\vec{x}$ is an eigenvector of A with eigenvalue λ then $\vec{x}$ is also an eigenvector of A^{-1} with eigenvalue λ^{-1}.

Suppose that $\vec{x}$ is an eigenvector of A with eigenvalue λ. Then $\vec{x} \neq \vec{0}$ and $A\vec{x} = \lambda\vec{x}$. Since A is invertible, A^{-1} exists and multiplying by A^{-1} on both sides gives

$$A^{-1}A\vec{x} = A^{-1}(\lambda\vec{x})$$

or

$$I\vec{x} = \lambda A^{-1}\vec{x}$$

or

$$\vec{x} = \lambda(A^{-1}\vec{x}).$$

Now since A is invertible and λ is an eigenvalue, $\lambda \neq 0$ and so we can divide by λ. Thus, $A^{-1}\vec{x} = \lambda^{-1}\vec{x}$. Then, since $\vec{x} \neq \vec{0}$, we have that $\vec{x}$ is an eigenvector of A^{-1} with eigenvalue λ^{-1}.

Characteristic Polynomial & Finding Eigenvalues: The characteristic polynomial of A is $\det(A - \lambda I)$. The eigenvalues of A are the roots of the characteristic polynomial.

$$A\vec{v} = \lambda \vec{v},\ \vec{v} \neq \vec{0} \quad \text{iff} \quad A\vec{v} - \lambda\vec{v} = \vec{0},\ \vec{v} \neq \vec{0}$$
$$\text{iff} \quad A\vec{v} - \lambda I \vec{v} = \vec{0},\ \vec{v} \neq \vec{0}$$
$$\text{iff} \quad (A - \lambda I)\vec{v} = \vec{0},\ \vec{v} \neq \vec{0}$$
$$\text{iff} \quad \vec{v} \in \mathrm{Ker}(A - \lambda I),\ \vec{v} \neq \vec{0}$$
$$\text{iff} \quad \mathrm{Ker}(A - \lambda I) \neq \{\vec{0}\}$$
$$\text{iff} \quad \det(A - \lambda I) \neq 0.$$

Ex. 3. Find the eigenvalues of the matrix $A = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$.

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & -2 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & -1-\lambda \end{vmatrix} = -0 + (1-\lambda)\begin{vmatrix} 2-\lambda & -2 \\ 1 & -1-\lambda \end{vmatrix} - 0$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$= (1-\lambda)\left[(2-\lambda)(-1-\lambda) + 2\right]$$
$$= (1-\lambda)\left[-2 - 2\lambda + \lambda + \lambda^2 + 2\right]$$
$$= (1-\lambda)(\lambda^2 - \lambda)$$
$$= (1-\lambda)\lambda(\lambda - 1) = 0 \implies \lambda = 0, 1$$

Eigenspaces & Finding Eigenvectors: The eigenspace E_λ of an eigenvalue λ is the nullspace $N(A - \lambda I)$ of the matrix $A - \lambda I$. The eigenvectors of A with eigenvalue λ are the nonzero elements of E_λ .

Ex. 4. For a basis for the eigenspace of each eigenvalue of the matrix $A = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$.

$\underline{\lambda = 0}$: we want $(A - 0I)\vec{v} = \vec{0}$

$$\left[\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right] \xrightarrow{R1 - 2R3 \to R3} \left[\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \xrightarrow[R2 + R3 \to R3]{R1 - R2 \to R1} \left[\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\implies 2v_1 - 2v_3 = 0$, $v_2 = 0$ $\implies$ $v_1 = v_3$, $v_2 = 0$, v_3 free

So a basis of E_0 is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

$\dim(E_0) = 1$

$\underline{\lambda = 1}$: we want $(A - I)\vec{v} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \xrightarrow{R1 - R3 \to R3} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \begin{array}{l} v_1 + v_2 - 2v_3 = 0 \\ v_1 = -v_2 + 2v_3 \\ v_2, v_3 \text{ free} \end{array}$$

So a basis of E_1 is $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\}$.

$\dim(E_1) = 2$

From above, char. poly. is $(1-\lambda)\lambda(\lambda-1) = -\lambda(\lambda-1)^2$ alg. mult. $(0)=1$ geom. mult $(0)=1$
alg. mult. $(1)=2$ geom mult $(1)=2$

Algebraic Multiplicity: The algebraic multiplicity of λ is the number of times it appears as a root of the characteristic polynomial.

Geometric Multiplicity: The geometric multiplicity of λ is the dimension $\dim(E_\lambda)$ of the eigenspace E_λ.

Theorem: For any eigenvalue λ, $1 \leq$ (geometric multiplicity of λ) $\leq$ (algebraic multiplicity of λ).

Diagonalizability: Write down what it means to say that an $n \times n$ matrix A is diagonalizable.

Either (2) or (3) depending on the text. These statements are equivalent.
(study guide says 2, text says 3)

Theorem: Let A be an $n \times n$ matrix. The following are equivalent.

(1) A is diagonalizable.
(2) There is an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.
(3) There is a basis of $\mathbb{R}^n$ consisting of eigenvectors of A (i.e., an eigenbasis).
(4) The characteristic polynomial has n real roots (possibly repeated) and for each root λ,

$$\text{(geometric multiplicity of } \lambda) = \text{(algebraic multiplicity of } \lambda).$$

How to decide if A is diag'ble or not

In this case, the basis of eigenvectors are the columns of P and the corresponding eigenvalues are the diagonal entries in the corresponding columns of D (i.e., in the same order).

Remark: This comes from the fact that if T is the linear map $T(x) = Ax$ and α is an eigenbasis, then $D = [T]_\alpha^\alpha$ is the matrix of T with respect to α and $P = [I]_\alpha^{std}$ is the change of basis matrix from the basis α to standard coordinates on $\mathbb{R}^n$.

Say $\alpha = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an eigenbasis corresponding to e-vals $\lambda_1, \lambda_2, \lambda_3$.
Then $T(\vec{v}_i) = A\vec{v}_i = \lambda_i \vec{v}_i \Rightarrow [T(v_i)]_\alpha = \lambda_i \vec{e}_i$. Hence $[T]_\alpha^\alpha = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = D$
and for $[I]_\alpha^{std} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = P$ we have $[T]_{std}^{std} = [I]_\alpha^{std} [T]_\alpha^\alpha [I]_{std}^\alpha$
or $A = PDP^{-1} \Rightarrow P^{-1}AP = D.$

Note: It follows that if A has n *distinct* real eigenvalues, then A is diagonalizable. However, if A has repeated eigenvalues, it may or may not be diagonalizable.
Then alg.mu.$(\lambda) = 1 =$ geom.mu(λ) $\forall \lambda$

Ex. 5. Determine whether or not the matrix A from examples 3 and 4 is diagonalizable. If it is, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

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In 3 and 4 we found that the eigenvalues of A are $\lambda = 0$ and $\lambda = 1$ are both real. And algebraic multiplicity $(0) = 1 =$ geometric multiplicity (0) and algebraic multiplicity $(1) = 2 =$ geometric multiplicity (1). Thus, A is diagonalizable. A basis for E_0 is $\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$ and a basis for E_1 is $\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}\right\}$. So for $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, we have

$P^{-1}AP = D.$

Ex. 6. Let the matrix A be as defined below. Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$, or show that no such matrices exist.

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \qquad A = \begin{bmatrix} 0 & 4 & 2 \\ 0 & -2 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

1. Find e-vals: $\det(A - \lambda I) = \begin{vmatrix} -\lambda & 4 & 2 \\ 0 & -2-\lambda & 1 \\ 0 & -1 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -2-\lambda & 1 \\ -1 & -\lambda \end{vmatrix} - 0 + 0$

$$= -\lambda\left[(-2-\lambda)(-\lambda) + 1\right]$$

$$= -\lambda(\lambda^2 + 2\lambda + 1)$$

$$= -\lambda(\lambda+1)(\lambda+1) \Rightarrow \lambda = 0, -1$$

2. Find e-vecs: Since alg. mu.$(0) = 1 =$ geom. mu.(0) automatically, diagonalizability depends on geom. mu.(-1). So, let's start with $\lambda = -1$.

$\underline{\lambda = -1}$: We want $(A + I)\vec{v} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow[R2 - R3 \to R3]{4R2 + R1 \to R1} \left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} v_1 = -6v_3 \\ v_2 = v_3 \\ v_3 \text{ free} \end{array}$$

So, $\dim(E_{-1}) = 1$. Since the algebraic multiplicity of $\lambda = -1$ is 2, the alg. mult. of -1 is not equal to its geom. mult. Thus, A is <u>not</u> diagonalizable, and so no such matrices exist.

Additional Problems

Ex. 7. Consider the matrix $A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 0 & -5 \\ 1 & 0 & 0 \end{bmatrix}$.

(a) Find all eigenvalues of A. $\lambda = 1, 0$

(b) Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$, or show that no such matrices exist.

Ex. 8. Let A be the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ -6 & -1 & -3 \end{bmatrix}$. $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $P = \begin{pmatrix} -1 & -1 & -2 \\ 0 & 0 & -3 \\ 2 & 3 & 3 \end{pmatrix}$

(a) Find all eigenvalues of A.

(b) Find a a basis of each eigenspace.

(c) Find a diagonal matrix D and an invertible matrix P such that $D = PAP^{-1}$, or show that no such matrices exist.

e-vals are $\lambda = 0, 0, 5$

Ex. 9. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix}.$$

eigenbasis = $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right\}$

Find a basis for $\mathbb{R}^3$ consisting of eigenvectors of A, or else prove that there is no such basis.

Ex. 10. Consider the matrix

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 5 & 0 \\ 1 & 2 & 2 \end{bmatrix}.$$

e-vals = 2, 3, 3
but geom mu(3) = 1

Is A diagonalizable? Why or why not?

Ex. 11. Let V be a finite-dimensional vector space, and let $T : V \to V$ be a linear transformation. Prove that 0 is an eigenvalue of T if and only if the image (i.e., range) of T is *not* equal to V. also on similar to Ex. 1, 2018 exam

Ex. 12. Suppose that a 3×3 matrix A has 0 as an eigenvalue.

(a) What are the possible values of the rank of A? Justify your answer. 0, 1, or 2

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by $T(x) = Ax$. Can T possibly be one-to-one? no Can T be onto? no Justify your answer.

Ex. 13. Let A, B be $n \times n$ matrices that commute, i.e. $AB = BA$. Let $v \in \mathbb{R}^n$ be an eigenvector of A such that $Bv \neq 0$. Prove that Bv is also an eigenvector of A. check your HW: 4.2 #14

Ex. 14. Prove that if $T : V \to V$ is a linear map then the eigenspace E_0 corresponding to the eigenvalue $\lambda = 0$ is equal to $\mathrm{Ker}(T)$. Use the def.

Ex. 15. Suppose that A is an $n \times n$ matrix that satisfies $A^2 = I$, where I is the $n \times n$ identity matrix. Show that if λ is an eigenvalue of A then $\lambda = 1$ or $\lambda = -1$. use same strategy as Ex. 2

Ex. 16. Let A be an $n \times n$ matrix and λ be an eigenvalue of A. Prove that λ^m is an eigenvalue of A^m for all integers $m \geq 1$. check your HW: 4.1 #13

Ex. 17. Let A be an $n \times n$ matrix and $\alpha \in \mathbb{R}$ be a scalar that is NOT an eigenvalue of A. Suppose that μ is an eigenvalue for the matrix $B = (A - \alpha I)^{-1}$ with corresponding eigenvector v. Prove that v is also an eigenvector for A and find a formula for the corresponding eigenvalue of A in terms of μ and α. on 2010 exam