4. Eigenvalues and Eigenvectors

(References: Comps Study Guide for Linear Algebra Section 4;
Damiano & Little, *A Course in Linear Algebra*, Chapter 4)

Let A be an $n \times n$ matrix, $\lambda \in \mathbb{R}$ be a scalar, and let $\vec{v} \in \mathbb{R}$. To say $\vec{v}$ is an eigenvector of A with eigenvalue λ means $\vec{v} \neq 0$ and $A\vec{v} = \lambda\vec{v}$.

Eigenvalues: Write down what it means to say that λ is an eigenvalue of A.

Eigenvector: Write down what it means to say that $\vec{v}$ is an eigenvector of A.

Ex. 1. Let A be an $n \times n$ matrix. Prove that A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A. (Note: This is a common theorem that you could usually use without proof.)

Ex. 2. Suppose that A is an invertible $n \times n$ matrix. Prove that if $\vec{x}$ is an eigenvector of A with eigenvalue λ then $\vec{x}$ is also an eigenvector of A^{-1} with eigenvalue λ^{-1}.

Characteristic Polynomial & Finding Eigenvalues: The characteristic polynomial of A is $\det(A - \lambda I)$. The eigenvalues of A are the roots of the characteristic polynomial.

Ex. 3. Find the eigenvalues of the matrix $A = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$.

Eigenspaces & Finding Eigenvectors: The eigenspace E_λ of an eigenvalue λ is the nullspace $N(A - \lambda I)$ of the matrix $A - \lambda I$. The eigenvectors of A with eigenvalue λ are the nonzero elements of E_λ .

Ex. 4. For a basis for the eigenspace of each eigenvalue of the matrix $A = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$.

Algebraic Multiplicity: The algebraic multiplicity of λ is the number of times it appears as a root of the characteristic polynomial.

Geometric Multiplicity: The geometric multiplicity of λ is the dimension $\dim(E_\lambda)$ of the eigenspace E_λ.

Theorem: For any eigenvalue λ, $1 \leq$ (geometric multiplicity of λ) $\leq$ (algebraic multiplicity of λ).

Diagonalizability: Write down what it means to say that an $n \times n$ matrix A is diagonalizable.

Theorem: Let A be an $n \times n$ matrix. The following are equivalent.

(1) A is diagonalizable.
(2) There is an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.
(3) There is a basis of $\mathbb{R}^n$ consisting of eigenvectors of A (i.e., an eigenbasis).
(4) The characteristic polynomial has n real roots (possibly repeated) and for each root λ,

$$(\text{geometric multiplicity of } \lambda) = (\text{algebraic multiplicity of } \lambda).$$

In this case, the basis of eigenvectors are the columns of P and the corresponding eigenvalues are the diagonal entries in the corresponding columns of D (i.e., in the same order).

Remark: This comes from the fact that if T is the linear map $T(x) = Ax$ and α is an eigenbasis, then $D = [T]_\alpha^\alpha$ is the matrix of T with respect to α and $P = [I]_\alpha^{std}$ is the change of basis matrix from the basis α to standard coordinates on $\mathbb{R}^n$.

Note: It follows that if A has n *distinct* real eigenvalues, then A is diagonalizable. However, if A has repeated eigenvalues, it may or may not be diagonalizable.

Ex. 5. Determine whether or not the matrix A from examples 3 and 4 is diagonalizable. If it is, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

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Ex. 6. Let the matrix A be as defined below. Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$, or show that no such matrices exist.

$$A = \begin{bmatrix} 0 & 4 & 2 \\ 0 & -2 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Additional Problems

Ex. 7. Consider the matrix $A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 0 & -5 \\ 1 & 0 & 0 \end{bmatrix}$.

(a) Find all eigenvalues of A.
(b) Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$, or show that no such matrices exist.

Ex. 8. Let A be the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ -6 & -1 & -3 \end{bmatrix}$.

(a) Find all eigenvalues of A.
(b) Find a a basis of each eigenspace.
(c) Find a diagonal matrix D and an invertible matrix P such that $D = PAP^{-1}$, or show that no such matrices exist.

Ex. 9. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix}.$$

Find a basis for $\mathbb{R}^3$ consisting of eigenvectors of A, or else prove that there is no such basis.

Ex. 10. Consider the matrix

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 5 & 0 \\ 1 & 2 & 2 \end{bmatrix}.$$

Is A diagonalizable? Why or why not?

Ex. 11. Let V be a finite-dimensional vector space, and let $T : V \to V$ be a linear transformation. Prove that 0 is an eigenvalue of T if and only if the image (i.e., range) of T is *not* equal to V.

Ex. 12. Suppose that a 3×3 matrix A has 0 as an eigenvalue.

(a) What are the possible values of the rank of A? Justify your answer.
(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by $T(x) = Ax$. Can T possibly be one-to-one? Can T be onto? Justify your answer.

Ex. 13. Let A, B be $n \times n$ matrices that commute, i.e. $AB = BA$. Let $v \in \mathbb{R}^n$ be an eigenvector of A such that $Bv \neq 0$. Prove that Bv is also an eigenvector of A.

Ex. 14. Prove that if $T : V \to V$ is a linear map then the eigenspace E_0 corresponding to the eigenvalue $\lambda = 0$ is equal to $\mathrm{Ker}(T)$.

Ex. 15. Suppose that A is an $n \times n$ matrix that satisfies $A^2 = I$, where I is the $n \times n$ identity matrix. Show that if λ is an eigenvalue of A then $\lambda = 1$ or $\lambda = -1$.

Ex. 16. Let A be an $n \times n$ matrix and λ be an eigenvalue of A. Prove that λ^m is an eigenvalue of A^m for all integers $m \geq 1$.

Ex. 17. Let A be an $n \times n$ matrix and $\alpha \in \mathbb{R}$ be a scalar that is NOT an eigenvalue of A. Suppose that μ is an eigenvalue for the matrix $B = (A - \alpha I)^{-1}$ with corresponding eigenvector v. Prove that v is also an eigenvector for A and find a formula for the corresponding eigenvalue of A in terms of μ and α.