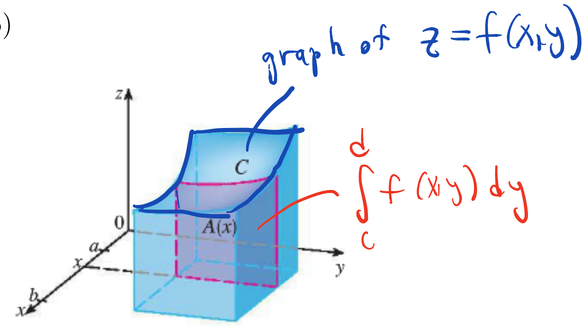
4. Double Integrals

(from Stewart, *Calculus*, Chapter 15)

Partial Integrals: $\int_c^d f(x,y)\ dy$ is calculated by holding x constant and integrating with respect to y from $y=c$ to $y=d$.

Note that the result is a function of x.

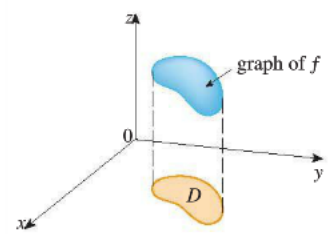
For each fixed x, the trace of $f(x,y)$ is a curve C in the plane $x=$ constant. The partial integral $A(x)=\int_c^d f(x,y)\ dy$ is the area under the curve C from $y=c$ to $y=d$.



Similarly, $\int_a^b f(x,y)\ dx$ is calculated by holding y constant and integrating with respect to x from $x=a$ to $x=b$. The result is a function of y.

Finding Volume with a Double Integral:

$\iint\limits_D f(x,y)dA$ is the signed volume between the surface $z=f(x,y)$ and the region D in the xy-plane.



Finding Area with a Double Integral:

$\text{Area}(D)=\iint\limits_D 1\ dA$

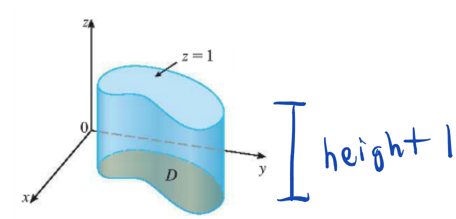


FIGURE 19
Cylinder with base D and height 1

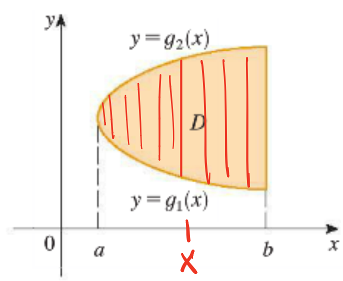
Iterated Integrals

Vertically Simple Regions:

$D=\{(x,y)\ :\ a\le x\le b,\ g_1(x)\le y\le g_2(x)\}$

$$\iint_D f(x,y)\ dA=\int_a^b\left(\int_{g_1(x)}^{g_2(x)} f(x,y)\ dy\right)dx$$

dy

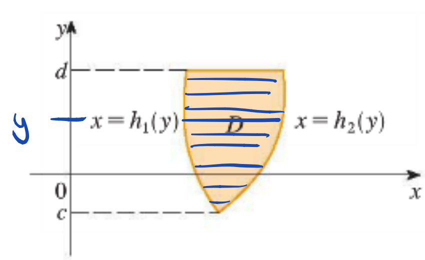


Horizontally Simple Regions:

$D=\{(x,y)\ :\ c\le y\le d,\ h_1(y)\le x\le h_2(y)\}$

$$\iint_D f(x,y)\ dA=\int_c^d\left(\int_{h_1(y)}^{h_2(y)} f(x,y)\ dx\right)dy$$

dx



Ex. 1. Find the volume of the solid that lies under the surface $z = xy$ and above the region D in the xy-plane bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$. $\Rightarrow$ $x = \frac{1}{2}y^2 - 3$

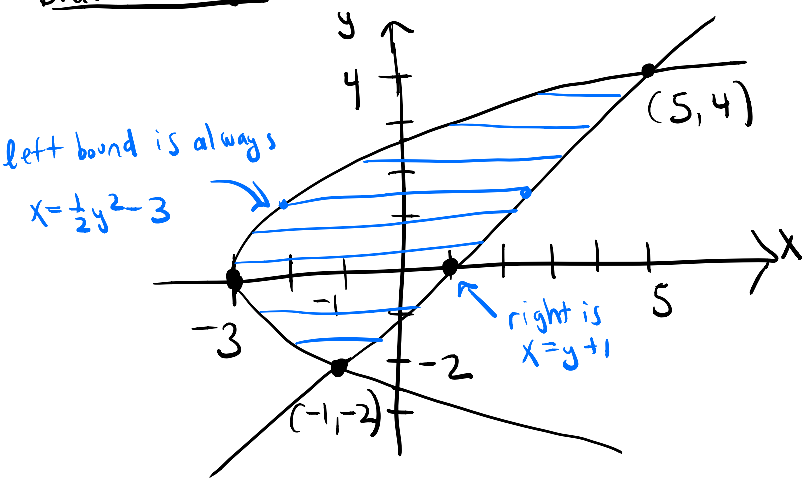
$x = y + 1$

$$V = \iint_D xy \, dA$$

<u>Intersection</u>

$$y + 1 = \frac{1}{2}y^2 - 3$$
$$y^2 - 2y - 8 = 0$$
$$(y-4)(y+2) = 0$$
$$y = 4, \quad y = -2$$
$$x = 5 \qquad x = -1$$

<u>Draw the Region</u> (Always draw D!)



$$\iint_D xy \, dA = \int_{-2}^{4} \left(\int_{\frac{1}{2}y^2-3}^{y+1} xy \, dx \right) dy$$

$$= \int_{-2}^{4} \frac{1}{2}x^2 y \Big|_{x=\frac{1}{2}y^2-3}^{x=y+1} dy$$

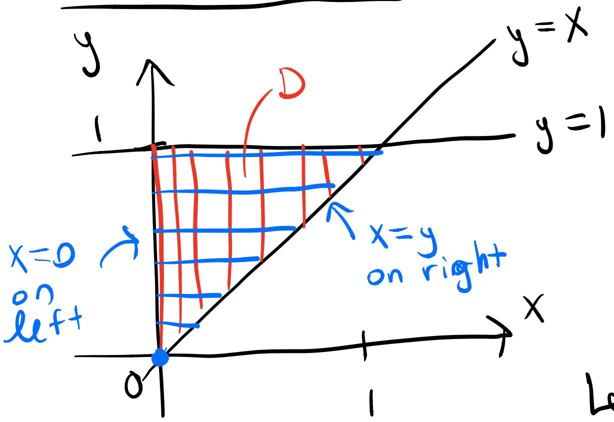
$$= \int_{-2}^{4} \frac{1}{2}\left[(y+1)^2 y - \left(\frac{1}{2}y^2 - 3\right)^2 y\right] dy$$

$$= \cdots = 36$$

Ex. 2. Evaluate $\int_0^1 \left(\int_{x = y}^{1 = y} \sin(y^2) \, dy \right) dx$.

We can't compute $\int \sin(y^2) dy$ so we try changing the order of int.

<u>Draw the Region</u>



$$\int_0^1 \int_x^1 \sin(y^2) \, dy \, dx = \int_0^1 \left(\int_0^y \sin(y^2) \, dx \right) dy$$

$$= \int_0^1 \sin(y^2) x \Big|_{x=0}^{x=y} dy$$

$$= \int_{y=0}^{y=1} y \sin(y^2) \, dy$$

Let $u = y^2$
$du = 2y \, dy$

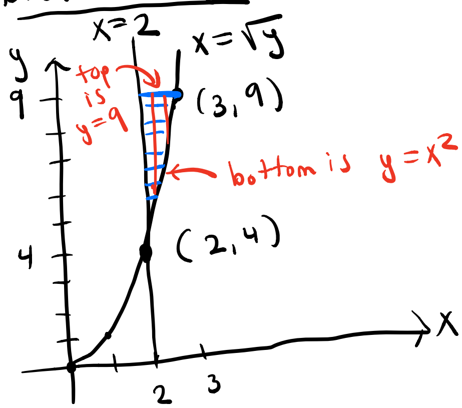
$u(0) = 0^2 = 0$
$u(1) = 1^2 = 1$

$$= \int_0^1 \frac{1}{2} \sin(u) \, du$$

$$= -\frac{1}{2}\cos(u)\Big|_0^1 = -\frac{1}{2}\cos(1) + \frac{1}{2}$$

Ex. 3. Reverse the order of integration in the integral $\int_4^9 \left(\int_{2 = x}^{\sqrt{y} = x} f(x,y)\, dx \right) dy$.

Draw the Region



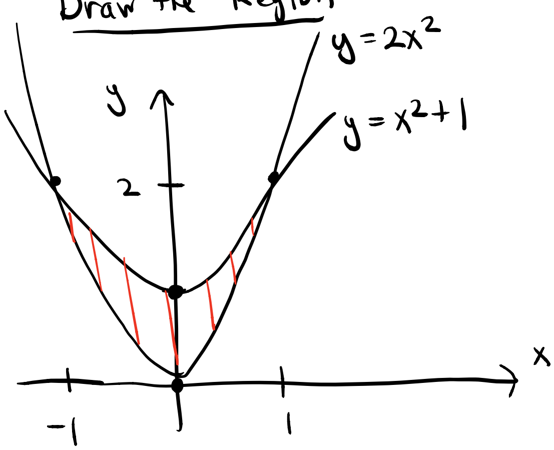
$$= \int_2^3 \left(\int_{x^2}^9 f(x,y)\, dy \right) dx$$

Ex. 4. Evaluate $\iint_D (x+2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Intersection $2x^2 = 1 + x^2$

$x^2 = 1 \Rightarrow x = \pm 1,\ y = 2$

Draw the Region



$$\iint_D (x+2y)\, dA = \int_{-1}^{1} \int_{2x^2}^{x^2+1} (x+2y)\, dy\, dx$$

$$= \int_{-1}^{1} \left(xy + y^2 \right) \Big|_{y=2x^2}^{y=x^2+1} dx$$

$$= \int_{-1}^{1} \left[x(x^2+1) + (x^2+1)^2 - 2x^3 - (2x^2)^2 \right] dx$$

$$= \int_{-1}^{1} \left(x^3 + x + x^4 + 2x^2 + 1 - 2x^3 - 4x^4 \right) dx$$

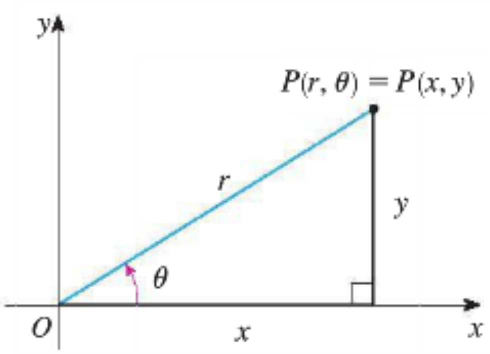
$$= \int_{-1}^{1} \left(-3x^4 - x^3 + 2x^2 + x + 1 \right) dx$$

$$= -\frac{3}{5}x^5 - \frac{x^4}{4} + \frac{2}{3}x^3 + \frac{x^2}{2} + x \Big|_{-1}^{1}$$

$$= -\frac{3}{5} + -\frac{3}{5} + \frac{2}{3} + \frac{2}{3} + 1 + 1$$

$$= -\frac{6}{5} + \frac{4}{3} + 2 = \frac{32}{15}$$

Polar Coordinates (r, θ)



r is the signed distance from the origin

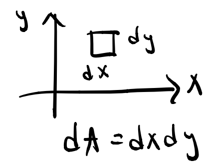
θ is the angle measured counter-clockwise from the positive x-axis

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

$$r^2 = x^2 + y^2$$

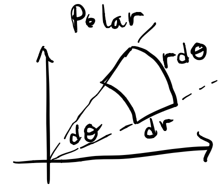
$$\tan(\theta) = \frac{y}{x}$$

Rectangular coords



$dA = dx\,dy$

Polar

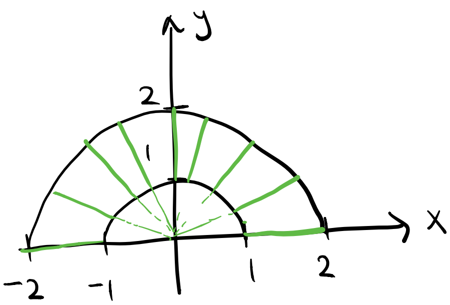


Double Integrals in Polar Coordinates:

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,\underline{r\,dr\,d\theta}.$$

Ex. 5. Evaluate $\iint_R (3x + 4y^2)dA$ where R is the region in the upper half plane $x \geq 0$ bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Draw the Region



$$\iint_R (3x + 4y^2)\,dA = \int_0^\pi \int_1^2 \left[3r\cos\theta + 4(r\sin\theta)^2\right] r\,dr\,d\theta$$

$$= \int_0^\pi \int_1^2 \left[3r^2\cos\theta + 4r^3\sin^2\theta\right] dr\,d\theta$$

$$= \int_0^\pi r^3\cos\theta + r^4\sin^2\theta \Big|_{r=1}^{r=2} d\theta$$

$$= \int_0^\pi 7\cos\theta + 15\sin^2\theta \; d\theta$$

$$= \int_0^\pi 7\cos\theta + 15\left(\frac{1-\cos(2\theta)}{2}\right) d\theta$$

Trig Identity

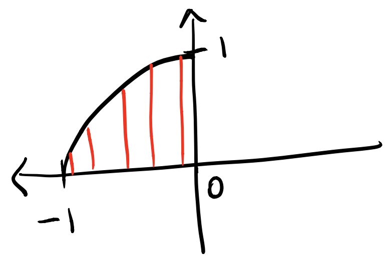
$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$= 7\sin\theta + \frac{15}{2}\theta - \frac{15\sin(2\theta)}{4} \Big|_0^\pi$$

$$= \frac{15}{2}\pi$$

Ex. 6. Evaluate the integral $\int_{-1}^{0}\left(\int_{0}^{\sqrt{1-x^2}=y} \cos(x^2+y^2)\, dy\right) dx$.

Draw the Region



$y=\sqrt{1-x^2} \Rightarrow y^2=1-x^2, \ y \geq 0$

$x^2+y^2=1$

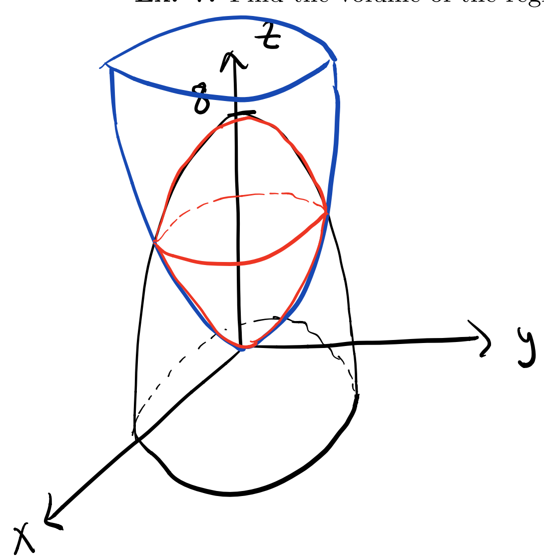
$$\int_{-1}^{0}\int_{0}^{\sqrt{1-x^2}} \cos(x^2+y^2)\,dy\,dx = \int_{\pi/2}^{\pi}\int_{0}^{1} \cos(r^2)\, r\,dr\,d\theta$$

$$= \int_{\pi/2}^{\pi} \frac{1}{2}\sin(r^2)\Big|_0^1 \,d\theta$$

$$= \frac{\pi}{4}\sin(1)$$

Ex. 7. Find the volume of the region bounded by the paraboloids $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$.

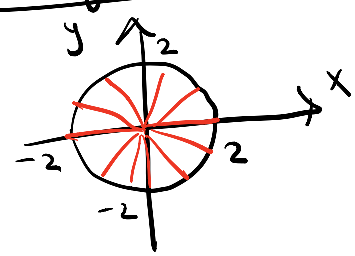
$z = -(x^2+y^2)+8$



Intersection $\quad x^2+y^2 = 8-x^2-y^2$

$x^2+y^2=4$

Region of Integration



$$\text{Volume} = \iint_D (8-x^2-y^2)-(x^2+y^2)\,dA$$

$$= \iint_D 8-2(x^2+y^2)\,dA$$

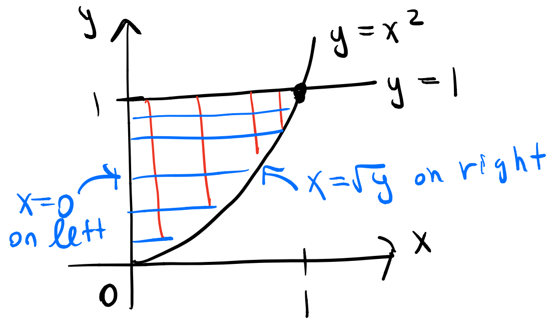
$$= \int_0^{2\pi}\int_0^{2}(8-2r^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} 4r^2 - \frac{2}{4}r^4\Big|_0^2\,d\theta \quad = (16-8)\,2\pi \quad = 16\pi$$

Ex. 8. Evaluate the integrals.

(a) $\displaystyle\int_0^1\left(\int_{x^2}^1 x^3\sin(y^3)\,dy\right)dx = \int_0^1\left(\int_0^{\sqrt{y}} x^3\sin(y^3)\,dx\right)dy$

Draw the Region



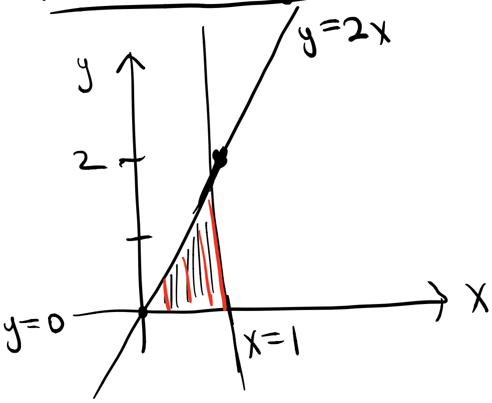
$$= \int_0^1\left(\frac{1}{4}x^4\Big|_0^{\sqrt{y}}\sin(y^3)\right)dy$$

$$= \int_0^1 \frac{1}{4}y^2\sin(y^3)\,dy$$

$$= -\frac{1}{12}\cos(y^3)\Big|_0^1 = -\frac{1}{12}\cos(1)+\frac{1}{12}$$

(b) $\displaystyle\iint_D e^{x^2}\,dA$ where D is the region bounded by the lines $y = 2x$, $y = 0$, and $x = 1$.

Draw the Region



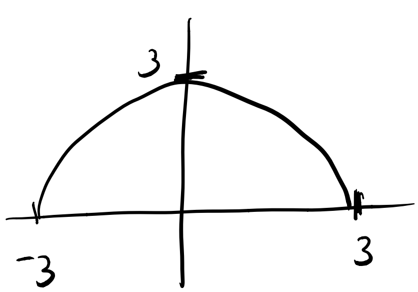
$$\iint_D e^{x^2}\,dA = \int_0^1\int_0^{2x} e^{x^2}\,dy\,dx$$

$$= \int_0^1 y\Big|_0^{2x} e^{x^2}\,dx$$

$$= \int_0^1 2xe^{x^2}\,dx$$

$$= e^{x^2}\Big|_0^1 = e-1$$

(c) $\displaystyle\iint_D e^{x^2+y^2}\,dA$ where D is the top half of the disk centered at the origin of radius 3.



$$\iint_D e^{x^2+y^2}\,dA = \int_0^{\pi}\int_0^3 e^{r^2}\,r\,dr\,d\theta$$

$$= \int_0^{\pi}\frac{1}{2}e^{r^2}\Big|_0^3\,d\theta$$

$$= \frac{\pi}{2}(e^9-1)$$