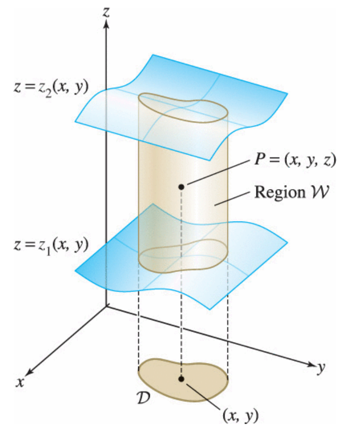
5. Triple Integrals
(from Stewart, *Calculus*, Chapter 15)



z-Simple Regions

$W = \{(x, y) \in D, \ \ z_1(x, y) \leq z \leq z_2(x, y)\}$

$\iiint_W f(x, y, z)\, dV = \iint_D \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\, dz\, dA$

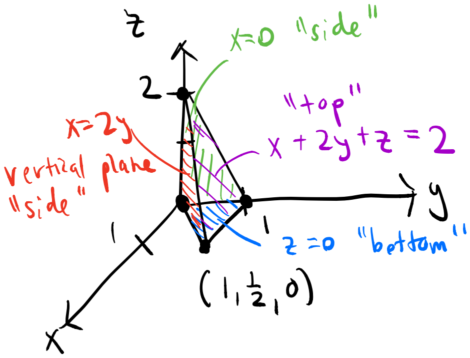
Finding Volume with a Triple Integral:

$\text{Volume}(W) = \iiint_W 1\, dV$.

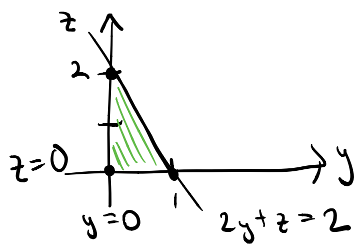
Ex. 1. Use a triple integral to find the volume of the tetrahedron bounded by the four planes $x = 0$, $z = 0$, $x = 2y$, and $x + 2y + z = 2$.

yz-plane xy-plane

Sketch the Region



In yz-plane, x=0

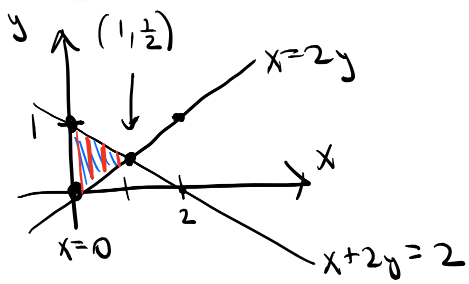


$x=0, \ x=2y \Rightarrow y=0$

$x=0, \ x+2y+z=2$

$\Rightarrow 2y+z=2$

In xy-plane, z=0



$z=0, \ x+2y+z=2$

$\Rightarrow x+2y=2$

$$V = \iiint_W 1\, dV = \int_0^1 \int_{\frac{1}{2}x}^{1-\frac{1}{2}x} \left(\int_0^{2-x-2y} 1 \, dz \right) dy\, dx$$

$$= \int_0^1 \int_{\frac{1}{2}x}^{1-\frac{1}{2}x} (2-x-2y)\, dy\, dx$$

$$= \int_0^1 (2-x)y - y^2 \Big|_{\frac{1}{2}x}^{1-\frac{1}{2}x} dx$$

$$= \int_0^1 (2-x)(1-x) - (1-\tfrac{1}{2}x)^2 + (\tfrac{1}{2}x)^2\, dx$$

$$= \int_0^1 (2-x)(1-x) + (x-1)\, dx$$

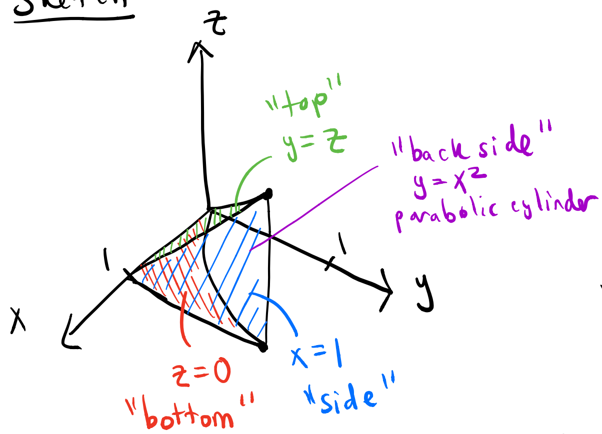
$$= \int_0^1 (x^2 - 2x + 1)\, dx$$

$$= \tfrac{1}{3}x^3 - x^2 + x \Big|_0^1$$

$$= \tfrac{1}{3}$$

Ex. 2. Reverse the order of integration in the integral $\displaystyle\int_0^1\int_0^{x^2}\left(\int_{0=z}^{y=z} f(x,y,z)\,dz\right)dy\,dx$. (Give all possibilities.)

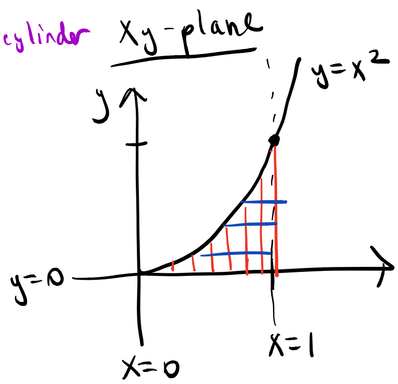
Sketch



"top" is the plane $y=z$

"bottom" is the xy-plane

xy-plane

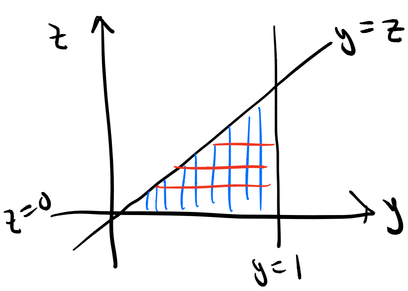


$0 \le y \le x^2$

$0 \le x \le 1$

$$\int_0^1\int_0^{x^2}\int_0^{y} f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_{\sqrt{y}}^1\left(\int_0^{y} f(x,y,z)\,dz\right)dx\,dy$$

yz-plane



$0 \le z \le y$

$0 \le y \le x^2$

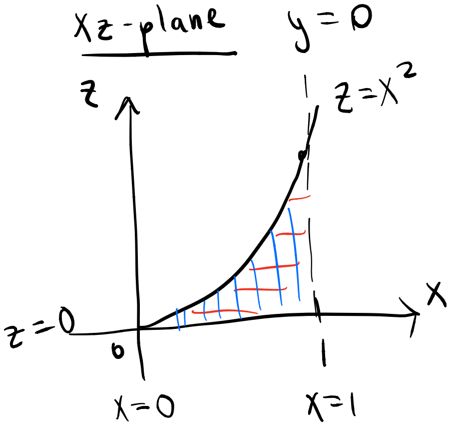
$0 \le x \le 1$

$\Rightarrow 0 \le y \le 1$

$$\int_0^1\int_z^1\left(\int_{\sqrt{y}}^1 f(x,y,z)\,dx\right)dy\,dz$$

$$= \int_0^1\int_0^{y}\left(\int_{\sqrt{y}}^1 f(x,y,z)\,dx\right)dz\,dy$$

xz-plane $y = 0$



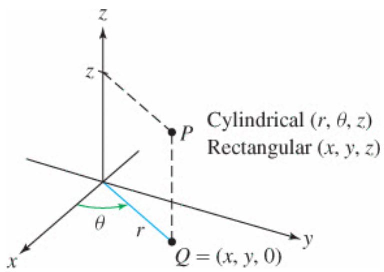
$y = z,\ y = x^2$

$\Rightarrow z = x^2$

$$\int_0^1\int_{\sqrt{z}}^1\left(\int_z^{x^2} f(x,y,z)\,dy\right)dx\,dz$$

$$= \int_0^1\int_0^{x^2}\left(\int_z^{x^2} f(x,y,z)\,dy\right)dz\,dx$$

Cylindrical Coordinates (r, θ, z)



(r, θ) are the polar coordinates of the projection (x, y) of the point (x, y, z) in the xy-plane, with $r \geq 0$

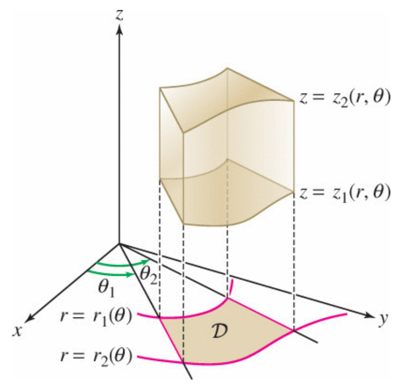
$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

$$r^2 = x^2 + y^2 \qquad \tan(\theta) = \frac{y}{x}$$

$$z = z$$

Triple Integrals in Cylindrical Coordinates:

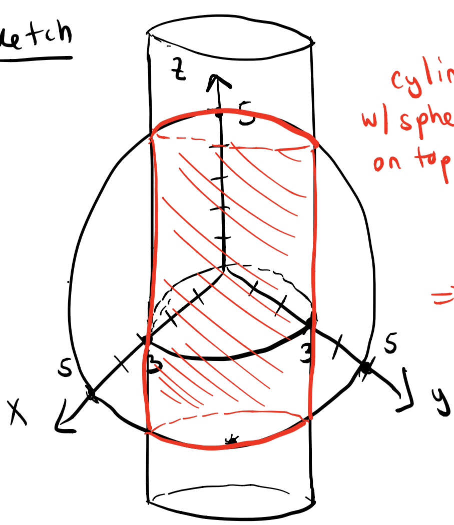
$$\iiint_W f(x, y, z)\, dV = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta.$$



Ex. 3. Find the volume of the region that is inside both the sphere $x^2 + y^2 + z^2 = 25$ and the cylinder $x^2 + y^2 = 9$.

radius 3   radius 5

Sketch



cylinder w/ sphere caps on top & bottom

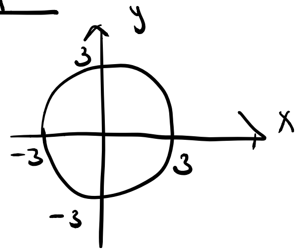
$x^2 + y^2 + z^2 = 25$

$\Rightarrow z = \pm\sqrt{25 - x^2 - y^2}$ are top & bottom surfaces

$\Rightarrow z = \sqrt{25 - r^2}$ on top

$z = \sqrt{25 - r^2}$ on bottom

xy-plane



circle $\Rightarrow$ polar coords

$$V = \iiint_W 1\, dV$$

$$= \int_0^{2\pi} \int_0^{3} \left(\int_{-\sqrt{25 - r^2}}^{\sqrt{25 - r^2}} 1\, r\, dz \right) dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^{3} 2\sqrt{25 - r^2}\, r\, dr\, d\theta$$

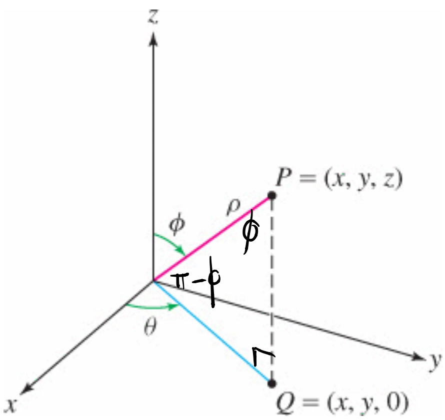
$$= 4\pi \left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)(25 - r^2)^{3/2} \Big|_0^3$$

$$= -\frac{4\pi}{3}\left(16^{3/2} - 25^{3/2}\right)$$

$$= -\frac{4\pi}{3}(64 - 125)$$

$$= \frac{244\pi}{3}$$

Spherical Coordinates (ρ, θ, ϕ)



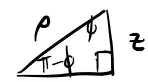
$\rho \geq 0$ is the distance from the origin

θ is the polar angle of the projection $(x, y, 0)$ in the xy-plane

ϕ is the angle of declination measured from the positive z-axis to $\overline{OP}$

$x = \rho \sin(\phi)\cos(\theta)$ $\quad$ $y = \rho\sin(\phi)\sin(\theta)$ $\quad$ $z = \rho\cos(\phi)$

$\rho^2 = x^2 + y^2 + z^2$ $\quad$ $\tan(\theta) = \dfrac{y}{x}$ $\quad$ $\cos(\phi) = \dfrac{z}{\rho}$



$\uparrow$ etc.

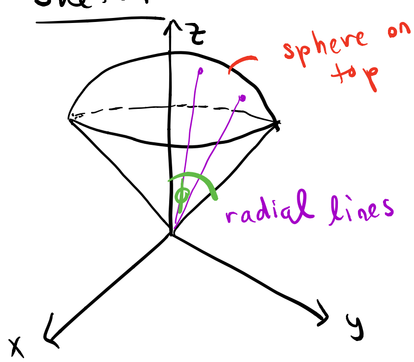
Triple Integrals in Spherical Coordinates:

$$\iiint_W f(x,y,z)\, dV = \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1(\theta,\phi)}^{\rho_2(\theta,\phi)} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2\sin\phi\, d\rho\, d\phi\, d\theta.$$

Ex. 4. Evaluate $\iiint_W z\, dV$ where W is the region above the cone $x^2 + y^2 = z^2$ and below the sphere $x^2 + y^2 + z^2 = 4$ for $z \geq 0$.

radius 2

Sketch



$$\iiint_W z\, dV = \int_0^{2\pi}\int_0^{\pi/4}\left(\int_0^2 \rho\cos\phi\,\rho^2\sin\phi\, d\rho\right) d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{1}{4}\rho^4\Big|_0^2 \cos\phi\sin\phi\, d\phi\, d\theta$$

$$= 2\pi \cdot 4\int_0^{\pi/4} \cos\phi\sin\phi\, d\phi$$

$$= 8\pi\int_0^{\pi/4}\frac{1}{2}\sin(2\phi)\, d\phi$$

Trig Id

$\sin(2x) = 2\sin x\cos x$

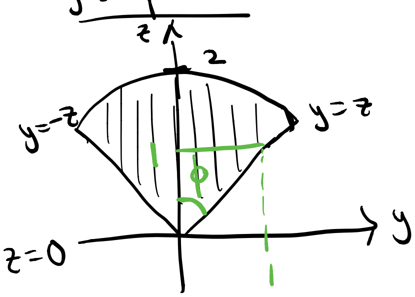
$$= 4\pi\left(-\frac{1}{2}\right)\cos(2\phi)\Big|_0^{\pi/4}$$

$$= -2\pi\left(\cos\left(\frac{\pi}{2}\right) - \cos(0)\right)$$

$$= 2\pi$$

For ϕ we need the cone angle.

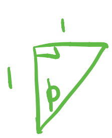
yz-plane $\quad x = 0$



$\Rightarrow y^2 = z^2$

$y = \pm z$

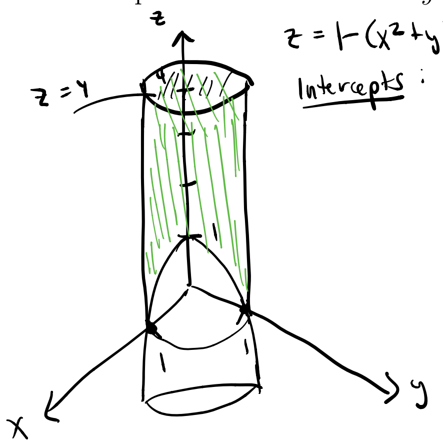
and $y^2 + z^2 = 4$



$\Rightarrow \tan\phi = 1$

$\Rightarrow \phi = \frac{\pi}{4}$

Ex. 5. A solid region W lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. Evaluate $\iiint_W \sqrt{x^2+y^2}\,dV$.



$z = 1-(x^2+y^2)$

Intercepts: $y = z = 0 \Rightarrow x = \pm 1$

$x = z = 0 \Rightarrow y = \pm 1$

$$\iiint_W \sqrt{x^2+y^2}\,dV = \int_0^{2\pi}\int_0^1\int_{1-r^2}^4 \sqrt{r^2}\; r\,dz\,dr\,d\theta$$

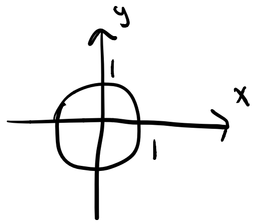
$$= \int_0^{2\pi}\int_0^1 r^2\left(4-(1-r^2)\right)dr\,d\theta$$

$$= 2\pi\int_0^1 \left(3r^2 + r^4\right)dr$$

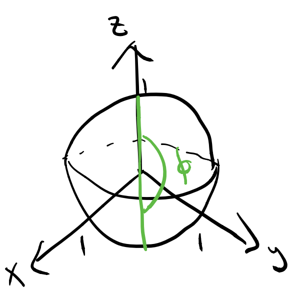
$$= 2\pi\left(r^3 + \frac{1}{5}r^4\right)\Big|_0^1$$

$$= \frac{12\pi}{5}$$

xy-plane



Ex. 6. Let $F(x, y, z) = x^2 + y^2 + z^2$. Compute $\iiint_W \|\nabla F\|\; dV$ where W is the region $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \le 1\}$.



$$\nabla F = \langle 2x, 2y, 2z\rangle$$

$$\|\nabla F\| = \sqrt{(2x)^2 + (2y)^2 + (2z)^2} = 2\sqrt{x^2+y^2+z^2}$$

$$\iiint_W \|\nabla F\|\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^1 2\sqrt{\rho^2}\,\rho^2 \sin\phi\; d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 2\rho^3 \sin\phi\; d\rho\, d\phi\, d\theta$$

$$= 2\pi\int_0^{\pi} \frac{1}{2}\rho^4\Big|_0^1 \sin\phi\; d\phi$$

$$= \pi\int_0^{\pi}\sin\phi\; d\phi$$

$$= -\pi\cos\phi\Big|_0^{\pi}$$

$$= 2\pi$$

Ex. 7. Evaluate the integral $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx$

Ex. 8. Evaluate the integral $\iiint_W xy\,dV$ where W is the region bounded by $z=9-x^2-y^2$, $z=0$, $y=x^2$, and $y=1$. and $y=0$.

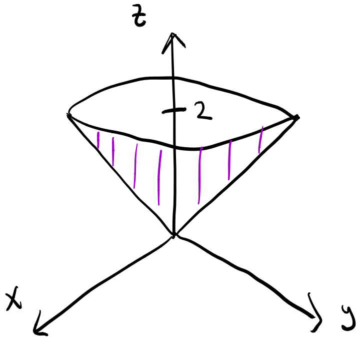
Ex. 9. Evaluate the integral $\iiint_W z\,dV$ where W is the solid tetrahedron bounded by the four planes $x=0$, $y=0$, $z=0$, and $x+y+z=1$.

Ex. 10. Evaluate the integral $\iiint_W e^{(x^2+y^2+z^2)^{3/2}}\,dV$ where W is the unit ball $x^2+y^2+z^2\le 1$.

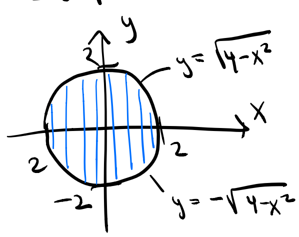
Ex. 7. Evaluate the integral $\int_{-2}^{2}\left(\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left(\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\right)dy\right)dx$

$z = \sqrt{x^2+y^2} \Rightarrow z^2 = x^2+y^2,\ z \geq 0$ upper half cone

$y = \pm\sqrt{4-x^2} \Rightarrow y^2 = 4+x^2 \Rightarrow x^2+y^2 = 4$ circle



xy-plane



$$\iiint_D (x^2+y^2)\,dA = \int_0^{2\pi}\int_0^{2}\int_r^{2} r^2\, r\, dz\, dr\, d\theta$$

$$= 2\pi\int_0^2 r^3(2-r)\,dr$$

$$= 2\pi\int_0^2 (2r^3 - r^4)\,dr$$

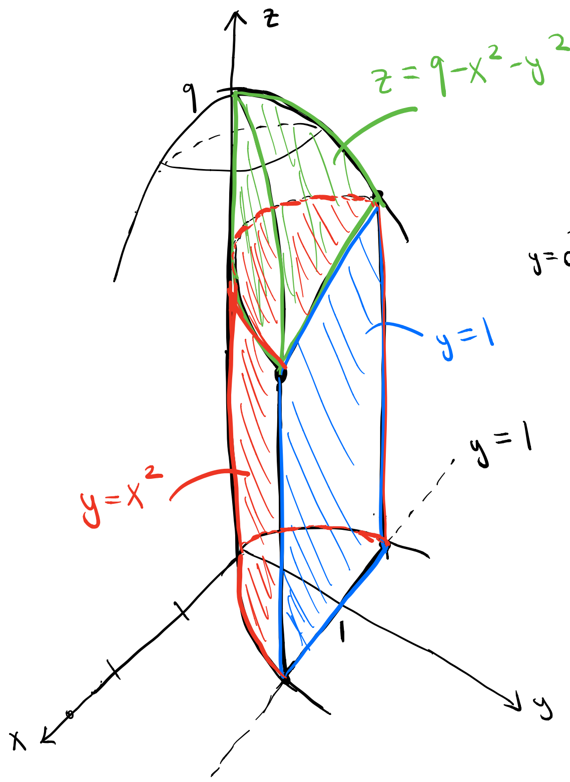
$$= 2\pi\left(\frac{1}{2}r^4 - \frac{1}{5}r^5\right)\Big|_0^2$$

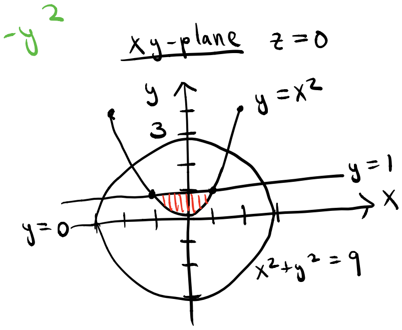
$$= 2\pi\left(8 - \frac{1}{5}\cdot 32\right)$$

$$= \frac{16\pi}{5}$$

Ex. 8. Evaluate the integral $\iiint_W xy\, dV$ where W is the region bounded by $z = 9 - x^2 - y^2$, $z = 0$, $y = x^2$, and $y = 1$. and $y = 0$.

$z = 9 - (x^2 + y^2)$





$x^2 + y^2 = 9$

$y = x^2$

$y = 1$

$$\iiint_W xy\, dV = \int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{9-x^2-y^2} xy\, dz\, dy\, dx$$

$$= \int_{-1}^{1}\int_{x^2}^{1} xy\left(9 - x^2 - y^2\right) dy\, dx$$

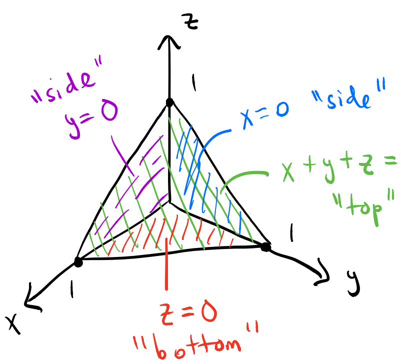
$$= \int_{-1}^{1}\int_{x^2}^{1} 9xy - x^3y - xy^3\, dy\, dx$$

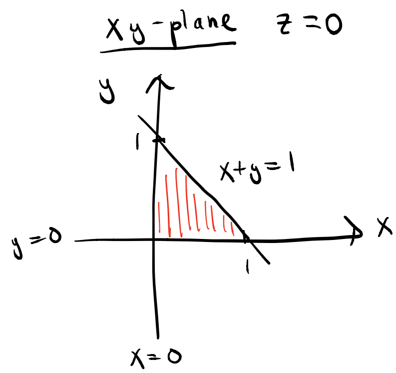
$$= \int_{-1}^{1} \frac{1}{2}\left(9x - x^3\right)y^2 - \frac{1}{4}xy^4 \Big|_{x^2}^{1} dx$$

$$= \int_{-1}^{1} \underbrace{\frac{1}{2}\underbrace{\left(9x - x^3\right)}_{\text{odd fn.}}\underbrace{\left(1 - x^4\right)}_{\text{even}}}_{\text{odd}} - \underbrace{\frac{1}{4}\underbrace{x}_{\text{odd}}\underbrace{\left(1 - x^8\right)}_{\text{even}}}_{\text{odd}}\, dx$$

$= 0$ since the integrand is odd and the integral is over $[-1, 1]$.

Ex. 9. Evaluate the integral $\iiint_W z\, dV$ where W is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.





$$\iiint_W z\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \frac{1}{2} z^2 \Big|_0^{1-x-y} dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \frac{1}{2} (1-x-y)^2\, dy\, dx$$

$$= \int_0^1 \frac{-1}{6} (1-x-y)^3 \Big|_0^{1-x} dx$$

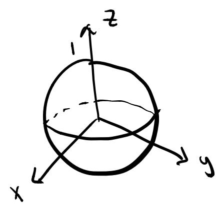
$$= \int_0^1 -\frac{1}{6} \left(0^3 - (1-x)^3 \right) dx$$

$$= \int_0^1 \frac{1}{6} (1-x)^3\, dx$$

$$= -\frac{1}{24} (1-x)^4 \Big|_0^1$$

$$= \frac{1}{24}$$

Ex. 10. Evaluate the integral $\iiint_W e^{(x^2+y^2+z^2)^{3/2}}\, dV$ where W is the unit ball $x^2+y^2+z^2 \le 1$.



$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} e^{(\rho^2)^{3/2}} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= 2\pi \int_0^{\pi}\int_0^{1} e^{\rho^3} \rho^2 \sin\phi \, d\rho\, d\phi$$

$$= 2\pi \int_0^{\pi} \frac{1}{3} e^{\rho^3} \Big|_0^1 \sin\phi \, d\phi$$

$$= 2\pi \int_0^{\pi} \frac{1}{3}(e-1) \sin\phi \, d\phi$$

$$= -\frac{2\pi}{3}(e-1) \cos\phi \Big|_0^{\pi}$$

$$= \frac{4\pi}{3}(e-1)$$