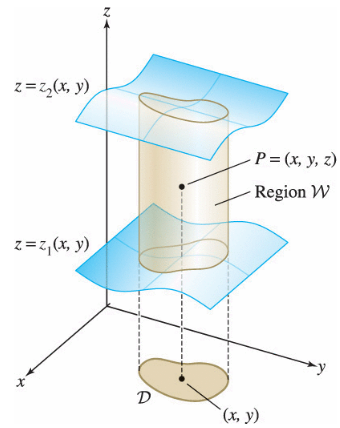
5. Triple Integrals
(from Stewart, *Calculus*, Chapter 15)



z-Simple Regions

$W = \{(x, y) \in D, \;\; z_1(x, y) \leq z \leq z_2(x, y)\}$

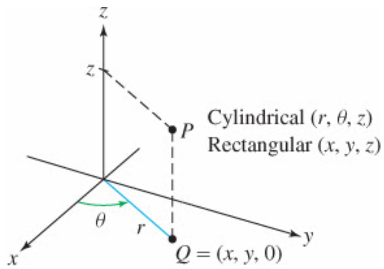
$\iiint_W f(x, y, z)\; dV = \iint_D \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\; dz\; dA$

Finding Volume with a Triple Integral:

$\text{Volume}(W) = \iiint_W 1\; dV.$

Ex. 1. Evaluate the integral $\displaystyle\iiint_W z\; dV$ where W is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

Cylindrical Coordinates (r, θ, z)



(r, θ) are the polar coordinates of the projection (x, y) of the point (x, y, z) in the xy-plane, with $r \geq 0$

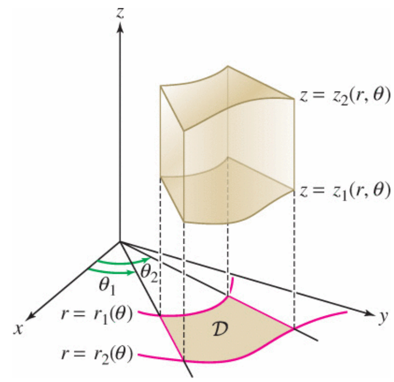
$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

$$r^2 = x^2 + y^2 \qquad \tan(\theta) = \frac{y}{x}$$

$$z = z$$

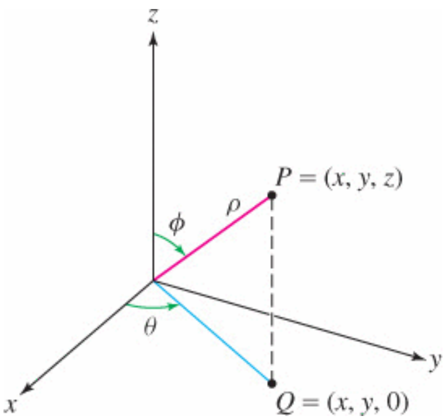
Triple Integrals in Cylindrical Coordinates:

$$\iiint_W f(x,y,z)\, dV = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta.$$



Ex. 2. Find the volume of the region that is inside both the sphere $x^2 + y^2 + z^2 = 25$ and the cylinder $x^2 + y^2 = 9$.

Spherical Coordinates (ρ, θ, ϕ)



$\rho \geq 0$ is the distance from the origin

θ is the polar angle of the projection $(x, y, 0)$ in the xy-plane

$0 \leq \phi \leq \pi$ is the angle of declination measured from the positive z-axis to $\overline{OP}$

$x = \rho \sin(\phi) \cos(\theta) \qquad y = \rho \sin(\phi) \sin(\theta) \qquad z = \rho \cos(\phi)$

$$\rho^2 = x^2 + y^2 + z^2 \qquad \tan(\theta) = \frac{y}{x} \qquad \cos(\phi) = \frac{z}{\rho}$$

Triple Integrals in Spherical Coordinates:

$$\iiint_W f(x, y, z)\, dV = \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \int_{\rho_1(\theta,\phi)}^{\rho_2(\theta,\phi)} f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

Ex. 3. Evaluate $\iiint_W z \, dV$ where W is the region above the cone $x^2 + y^2 = z^2$ and below the sphere $x^2 + y^2 + z^2 = 4$ for $z \geq 0$.

Ex. 4. A solid region W lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. Evaluate $\iiint_W \sqrt{x^2 + y^2} dV$.

Ex. 5. Let $F(x, y, z) = x^2 + y^2 + z^2$. Compute $\iiint_W \|\nabla F\| \, dV$ where W is the region $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$.

Ex. 6. Evaluate the integral $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2)\, dz\, dy\, dx$

Ex. 7. Evaluate the integral $\iiint_W xy\, dV$ where W is the region bounded by $z = 9 - x^2 - y^2$, $z = 0$, $y = x^2$, and $y = 1$ for $y \geq x^2$.

Ex. 8. Use a triple integral to find the volume of the tetrahedron bounded by the four planes $x = 0$, $z = 0$, $x = 2y$, and $x + 2y + z = 2$. (*Challenge.*)

Ex. 9. Evaluate the integral $\iiint_W e^{(x^2+y^2+z^2)^{3/2}}\, dV$ where W is the unit ball $x^2 + y^2 + z^2 \leq 1$.