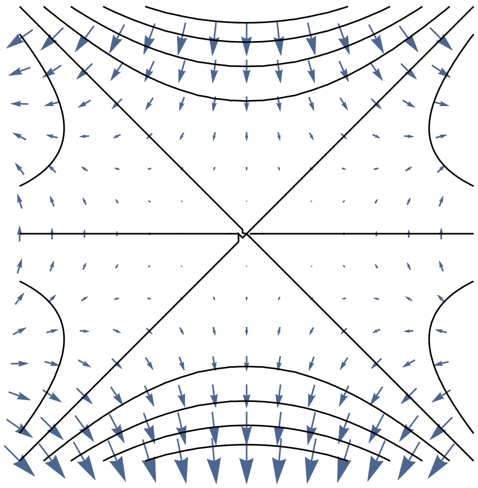
Multivariable Calculus: Review Session 4
6. Line Integrals
(from Stewart, *Calculus*, Chapter 16)

Vector Fields



A vector field is a function $\vec{F}$ that assigns a vector $\vec{F}$ to each point.

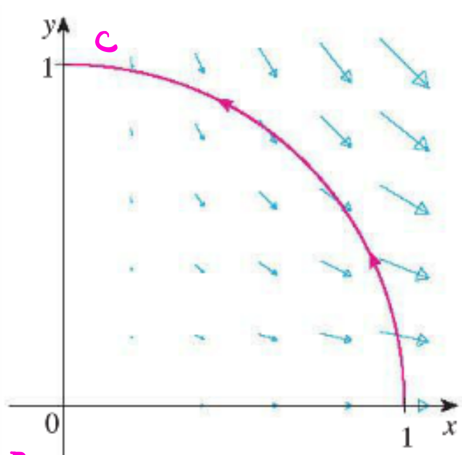
$$\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$$
$$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$$

Conservative / Gradient Vector Fields: $\vec{F} = \nabla f$ for some f. The function f is called a potential for $\vec{F}$.

Ex. $\vec{F}(x,y) = \langle 2xy, x^2 - 3y^2\rangle$ is a gradient vector field. A potential function is $f(x,y) = x^2y - y^3$.

A gradient vector field $\vec{F} = \nabla f$ is perpendicular to the level sets $f(x,y) = k$ of the potential.

Line Integrals of Vector Fields



Let $\vec{r}(t) = \langle x(t), y(t)\rangle$, $a \le t \le b$ be a parametrization of a curve C.

Line integral of $\vec{F}$ along C:

$$\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy$$

To compute:
$d\vec{r} = \vec{r}'(t)dt \qquad dx = x'(t)dt \qquad dy = y'(t)dt$

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\,dt$$

Rmk. Similarly for vector fields of three variables.

Interpretation: The line integral of $\vec{F}$ along C is the work performed by the force field in moving along C.

Ex. 1. Evaluate the line integral $\int_C x^2\,dx - xy\,dy$ where C is the arc $\vec{r}(t) = \langle\cos(t), \sin(t)\rangle$, $0 \le t \le \pi/2$.

$\vec{F}(x,y) = \langle x^2, -xy\rangle$

$x(t) = \cos t \qquad y(t) = \sin t$
$x'(t) = -\sin t \qquad y'(t) = \cos t$

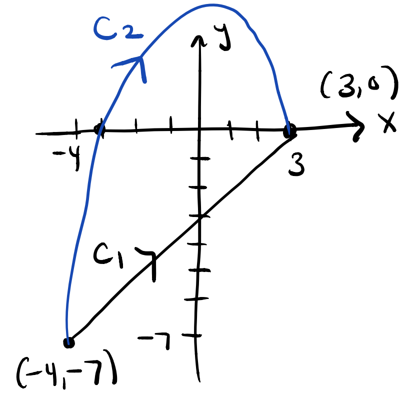
$$\int_C x^2dx - xy\,dy = \int_0^{\pi/2} \cos^2 t\,(-\sin t)dt - \cos t \sin t \cos t\,dt$$

$$= \int_0^{\pi/2} -2\cos^2 t \sin t\,dt$$

$u = \cos t \qquad u(0) = 1$
$du = -\sin t\,dt \qquad u(\pi/2) = 0$

$$= \int_1^0 2u^2\,du$$

$$= \frac{2}{3}u^3\Big|_1^0 = -\frac{2}{3}.$$



Ex. 2. Evaluate $\int_C y\,dx + x^2\,dy$ where
(a) $C = C_1$ is the line segment from $(-4,-7)$ to $(3,0)$.

Paramatrize C_1

$\vec{v} = \langle 3 - -4,\ 0 - -7\rangle = \langle 7, 7\rangle$

$\vec{r_0} = \langle -4, -7\rangle$

$x(t) = -4 + 7t \qquad y(t) = -7 + 7t, \quad 0 \le t \le 1$

$x'(t) = 7 \qquad y'(t) = 7$

$$\int_{C_1} y\,dx + x^2\,dy = \int_0^1 (-7+7t)\,7\,dt + (-4+7t)^2 \cdot 7\,dt$$

$$= 7\int_0^1 \left(-7 + 7t + 49t^2 - 56t + 16\right) dt$$

$$= 7\int_0^1 (49t^2 - 49t + 9)\,dt$$

$$= 7\left(\frac{49}{3}t^3 - \frac{49}{2}t^2 + 9t\right)\Big|_0^1$$

$$= 35/6$$

(b) $C = C_2$ is the arc of the parabola $y = 9 - x^2$ from $(-4,-7)$ to $(3,0)$.

Parametrize C_2

$x(t) = t \qquad y(t) = 9 - t^2, \quad -4 \le t \le 3$

$x'(t) = 1 \qquad y'(t) = -2t$

$$\int_{C_2} y\,dx + x^2\,dy = \int_{-4}^{3} (9-t^2)\cdot 1\,dt + t^2(-2t)\,dt$$

$$= \int_{-4}^{3} (-2t^3 - t^2 + 9)\,dt$$

$$= -\frac{1}{2}t^4 - \frac{1}{3}t^3 + 9t \Big|_{-4}^{3}$$

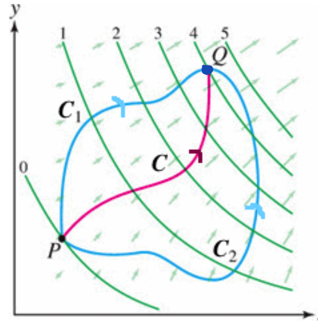
$$= \frac{721}{6}$$

The Fundamental Theorem for Line Integrals. Let C be a smooth (or piecewise smooth) curve given by the vector function $\vec{r}(t)$, $a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\vec{r} = f(\underline{\vec{r}(b)}) - f(\underline{\vec{r}(a)})$$

$\vec{r}(b)$ = terminal pt. $\vec{r}(a)$ = initial pt.

Independence of Path: By the fundamental theorem, if $\vec{F} = \nabla f$ is a gradient vector field, then the line integral depends only on the endpoints of C, not the path itself.



The figure at right shows the level sets of a function $f(x, y)$ and its gradient field. C, C_1, and C_2 are three paths from the point P to the point Q.

$$\int_C \nabla f \cdot d\vec{r} = \int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r} = f(Q) - f(P)$$

$$= 4 - 0 = 4$$

Ex. 3. Let $\vec{F}(x, y) = \langle 3 + 2xy, x^2 - 3y^2 \rangle$.

(a) Show that the line integral $\int_C \vec{F} \cdot d\vec{r}$ depends only on the endpoints of the path C and not on the path taken between those endpoints.

We show that $\vec{F} = \nabla f$ for some function $f = f(x,y)$. Since $\nabla f = \langle f_x, f_y \rangle$ we want $f_x(x,y) = 3 + 2xy$ and $f_y(x,y) = x^2 - 3y^2$. Integrating,

$\int f_x(x,y)\,dx = \int (3 + 2xy)\,dx$ and $\int f_y(x,y)\,dy = \int x^2 - 3y^2\,dy$

$f(x,y) = 3x + x^2 y + C(y)$ $\qquad$ $f(x,y) = x^2 y - y^3 + d(x)$.

Then for $f(x,y) = x^2 y + 3x - y^3$ we have $\nabla f(x,y) = \vec{F}(x,y)$. So by the FTLI,

$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(Q) - f(P)$ where P and Q are the initial & terminal points of C, respectively. Thus, the line integral depends only on the endpoints.

(b) Evaluate $\displaystyle\int_C \vec{F} \cdot d\vec{r}$ where C is the curve given by $\vec{r}(t) = \langle e^t \sin t, e^t \cos t \rangle$, $0 \le t \le \pi$.

$\vec{r}(0) = \langle 0, 1 \rangle$, $\vec{r}(\pi) = \langle 0, -e^{\pi} \rangle$

From a, $\int_C \vec{F} \cdot d\vec{r} = f(0, -e^{\pi}) - f(0,1)$
$= e^{3\pi} + 1$

Ex. 4. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y, z) = \langle yz, xz, xy + 2z \rangle$ and C is the line segment from $(1, 0, -2)$ to $(4, 6, 3)$.

Check if $\vec{F} = \nabla f$

$\int yz\,dx = xyz + c(y,z)$

$\int xz\,dy = xyz + d(x,z)$

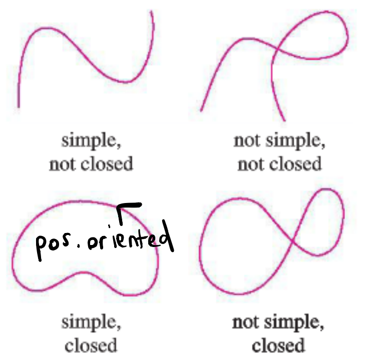
$\int (xy + 2z)\,dz = xyz + z^2 + \underbrace{g(x,y)}_{=0}$

Then for $f(x,y,z) = xyz + z^2$ we have $\nabla f = \vec{F}$.

So $\int_C \vec{F} \cdot d\vec{r} = f(4,6,3) - f(1,0,-2) = 72 + 9 - 4 = 77$.

Green's Theorem. Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. Let $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ be a smooth vector field on D. Then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA$$



Let C be a curve traversed once by $\vec{r}(t) = \langle x(t), y(t)\rangle$ for $a \le t \le b$.

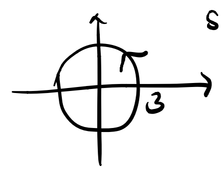
C is closed if the initial and terminal points of C coincide, i.e. $\vec{r}(a) = \vec{r}(b)$.

C is simple if C has no self-intersections except at the endpoints, i.e. $\vec{r}(t_1) \neq \vec{r}(t_2)$ for all $a < t_1 < t_2 < b$.

C is positively oriented if it is traversed in the counterclockwise direction as t increases from a to b. The region D is always to the left of C.

Rmk. The line integral of a vector field along a curve depends on the orientation of the curve as follows: If $-C$ denotes the curve C traversed in the opposite direction, then $\int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx$.

Ex. 5. Evaluate $\int_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$ where C is the circle $x^2+y^2=9$ traversed in the counterclockwise direction.



simple, smooth
pos. oriented
closed curve

$$\int_C \left(3y - e^{\sin x}\right) dx + \left(7x + \sqrt{y^4+1}\right) dy$$

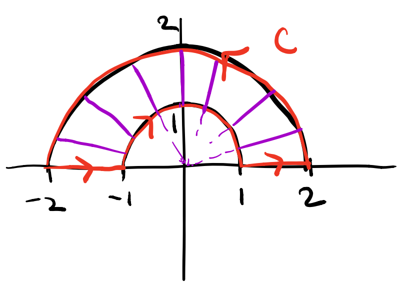
$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D (7-3)\,dA$$

$$= 4\,\text{Area}(D)$$

$$= 4\cdot 9\pi$$

$$= 36\pi$$

Ex. 6. Evaluate $\int_C y^2\,dx + 3xy\,dy$ where C is the boundary of the region between the circles $x^2+y^2=1$ and $x^2+y^2=4$ in the upper half plane, traversed counterclockwise.



simple, closed, ccw oriented
pw smooth

$$\int_C y^2\,dx + 3xy\,dy = \iint_D (3y - 2y)\,dA$$

$$= \int_0^{\pi}\int_1^{2} r\sin\theta\; r\,dr\,d\theta$$

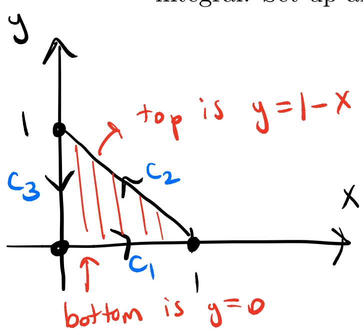
$$= \int_0^{\pi} \frac{1}{3}r^3\Big|_1^2 \sin\theta\,d\theta$$

$$= -\frac{7}{3}\cos\theta\Big|_0^{\pi}$$

$$= 14/3$$

Ex. 7. Let C be the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$, oriented counterclockwise and let $\vec{F}(x,y) = \langle x^3, xy\rangle$.

(a) According to Green's Theorem, the line integral $\int_C \vec{F}\cdot d\vec{r} = \int_C x^3\,dx + xy\,dy$ is equal to a certain double integral. Set up and evaluate this double integral.



$$\int_C \vec{F}\cdot d\vec{r} = \iint_D (y-0)\,dA$$

$$= \int_0^1\int_0^{1-x} y\,dy\,dx$$

$$= \int_0^1 \frac{1}{2}y^2\Big|_0^{1-x}\,dx$$

$$= \frac{1}{2}\int_0^1 (1-x)^2\,dx$$

$$= -\frac{1}{6}(1-x)^3\Big|_0^1$$

$$= \frac{1}{6}$$

(b) Verify Green's Theorem by evaluating the line integral directly.

$\vec{F} = \langle x^3, xy\rangle$

Parametrize the curves

C_1: $\vec{v}_1 = \langle 1,0\rangle - \langle 0,0\rangle = \langle 1,0\rangle$

$x = t$, $y = 0$, $0 \le t \le 1$

C_2: $\vec{v}_2 = \langle 0,1\rangle - \langle 1,0\rangle = \langle -1,1\rangle$

$x = 1-t$, $y = t$, $0 \le t \le 1$

C_3: $\vec{v}_3 = \langle 0,0\rangle - \langle 0,1\rangle = \langle 0,-1\rangle$

$x = 0$, $y = 1-t$, $0 \le t \le 1$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r}$$

$$= \int_0^1 t^3\cdot 1\,dt + t\cdot 0\cdot 0\,dt$$

$$+ \int_0^1 (1-t)^3(-1)\,dt + (1-t)t\cdot 1\,dt$$

$$+ \int_0^1 0^3\cdot 0\,dt + 0(1-t)(-1)\,dt$$

$$= \int_0^1 t^3 - (1-t)^3 + t - t^2\,dt$$

$$= \frac{1}{4}t^4 + \frac{1}{4}(1-t)^4 + \frac{1}{2}t^2 - \frac{1}{3}t^3\Big|_0^1$$

$$= \frac{1}{4} - \frac{1}{4} + \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6}.$$

Ex. 8. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y) = \langle xy^2, x^2y\rangle$ and C is traced out by $\vec{r}(t) = \langle \cos(t), 2\sin(t)\rangle$ for $0 \le t \le \frac{\pi}{2}$.

$\int xy^2\,dx = \frac{1}{2}x^2y^2 + c(y) \Rightarrow \vec{F} = \nabla f$ for $f(x,y) = \frac{1}{2}x^2y^2$.

$\int x^2y\,dy = \frac{1}{2}x^2y^2 + d(x)$

Then $\int_C \vec{F}\cdot d\vec{r} = f(\vec{r}(\pi/2)) - f(\vec{r}(0))$

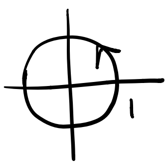
$= f(0,2) - f(1,0)$

$= 0.$

$\vec{r}(\pi/2) = \langle 0, 2\rangle$

$\vec{r}(0) = \langle 1, 0\rangle$

Ex. 9. Consider the vector field $\vec{F}(x,y) = \langle e^{3x} + xy, e^{3y} - xy\rangle$. Evaluate the integral $\int_C \vec{F}\cdot d\vec{r}$ where C is the unit circle $x^2+y^2 = 1$ oriented counterclockwise.



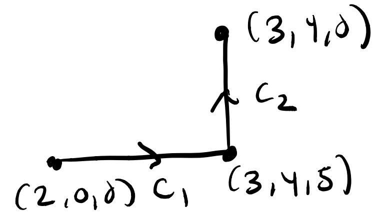
$\int_C \vec{F}\cdot d\vec{r} = \iint_D (-y - x)\,dA$

$= \int_0^{2\pi}\int_0^1 (-r\sin\theta - r\cos\theta)\,r\,dr\,d\theta$

$= \int_0^{2\pi} -\frac{1}{3}r^3\Big|_0^1 (\sin\theta + \cos\theta)\,d\theta$

$= \int_0^{2\pi} -\frac{1}{3}(\sin\theta + \cos\theta)\,d\theta$

$= 0.$



Ex. 10. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where C is the union of line segment C_1 from $(2,0,0)$ to $(3,4,5)$ followed by the vertical line segment C_2 from $(3,4,5)$ to $(3,4,0)$ for the following vector fields $\vec{F}$.

(a) $\vec{F}(x,y,z) = \langle y^2, 2xy, 4z\rangle$

$\int y^2\,dx = xy^2 + c(y,z)$

$\int 2xy\,dy = xy^2 + d(x,z)$

$\int 4z\,dz = 2z^2 + g(x,y)$

So $\vec{F} = \nabla f$ for $f(x,y,z) = xy^2 + 2z^2$.

Then $\int_C \vec{F}\cdot d\vec{r} = f(3,4,0) - f(2,0,0)$

$= 3\cdot 16 - 0$

$= 48$

(b) $\vec{F}(x,y,z) = \langle y, z, x\rangle$

$\int y\,dx = xy + c(y,z)$ problem!

$\int z\,dy = yz + d(x,z)$

$\int x\,dz = xz + h(x,y)$

Param. Curves C_1: $x = 2+t$, $y = 4t$, $z = 5t$, $0 \le t \le 1$

C_2: $x = 3$, $y = 4$, $z = 5 - 5t$, $0 \le t \le 1$

$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$

$= \int_0^1 4t(1) + 5t(4) + (2+t)5\,dt$

$+ \int_0^1 4(0) + (5-5t)0 + (3)(-5)\,dt$

$= \int_0^1 4t + 20t + 10 + 5t - 15\,dt$

$= \int_0^1 (29t - 5)\,dt = \frac{29}{2}t^2 - 5t\Big|_0^1 = \frac{19}{2}$