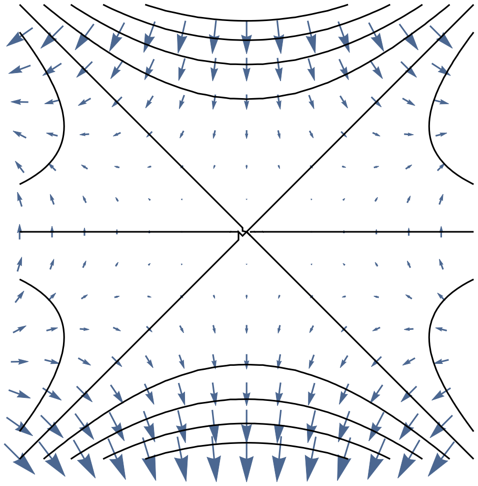
Multivariable Calculus: Review Session 4
6. Line Integrals
(from Stewart, *Calculus*, Chapter 16)

Vector Fields



A vector field is a function $\vec{F}$ that assigns a vector $\vec{F}$ to each point.

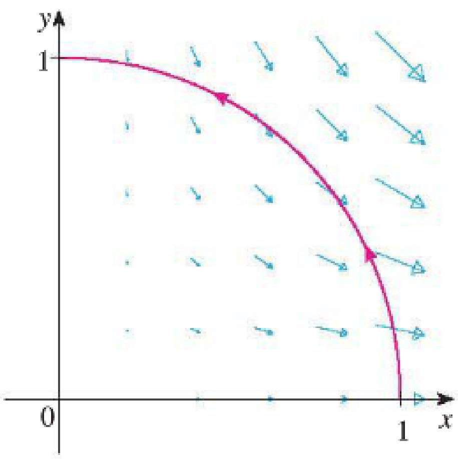
$$\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$$
$$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$$

Conservative / Gradient Vector Fields: $\vec{F} = \nabla f$ for some f. The function f is called a potential for $\vec{F}$.

Ex. $\vec{F}(x,y) = \langle 2xy, x^2 - 3y^2\rangle$ is a gradient vector field. A potential function is $f(x,y) = x^2y - y^3$.

A gradient vector field $\vec{F} = \nabla f$ is perpendicular to the level sets $f(x,y) = k$ of the potential.

Line Integrals of Vector Fields



Let $\vec{r}(t) = \langle x(t), y(t)\rangle$, $a \le t \le b$ be a parametrization of a curve C.

Line integral of $\vec{F}$ along C:

$$\int_C \vec{F}\cdot d\vec{r} = \int_C P\ dx + Q\ dy$$

To compute:
$d\vec{r} = \vec{r'}(t)dt \qquad dx = x'(t)dt \qquad dy = y'(t)dt$

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r'}(t)\ dt$$

Rmk. Similarly for vector fields of three variables.

Interpretation: The line integral of $\vec{F}$ along C is the work performed by the force field in moving along C.

Ex. 1. Evaluate the line integral $\int_C x^2\ dx - xy\ dy$ where C is the arc $\vec{r}(t) = \langle\cos(t), \sin(t)\rangle$, $0 \le t \le \pi/2$.

Ex. 2. Evaluate $\int_C y\,dx + x^2\,dy$ where
(a) $C = C_1$ is the line segment from $(-4,-7)$ to $(3,0)$.

(b) $C = C_2$ is the arc of the parabola $y = 9 - x^2$ from $(-4,-7)$ to $(3,0)$.

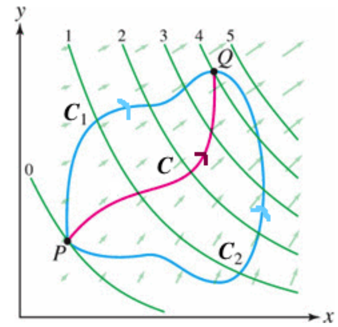
The Fundamental Theorem for Line Integrals. Let C be a smooth (or piecewise smooth) curve given by the vector function $\vec{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Independence of Path: By the fundamental theorem, if $\vec{F} = \nabla f$ is a gradient vector field, then the line integral depends only on the endpoints of C, not the path itself.

The figure at right shows the level sets of a function $f(x, y)$ and its gradient field. C, C_1, and C_2 are three paths from the point P to the point Q.

$$\int_C \nabla f \cdot d\vec{r} = \int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r} = f(Q) - f(P)$$



Ex. 3. Let $\vec{F}(x, y) = \langle 3 + 2xy, x^2 - 3y^2 \rangle$.

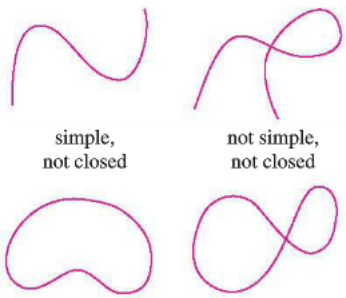
(a) Show that the line integral $\int_C \vec{F} \cdot d\vec{r}$ depends only on the endpoints of the path C and not on the path taken between those endpoints.

(b) Evaluate $\displaystyle\int_C \vec{F} \cdot d\vec{r}$ where C is the curve given by $\vec{r}(t) = \langle e^t \sin t, e^t \cos t \rangle$, $0 \leq t \leq \pi$.

Ex. 4. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y, z) = \langle yz, xz, xy + 2z \rangle$ and C is the line segment from $(1, 0, -2)$ to $(4, 6, 3)$.

Green's Theorem. Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. Let $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ be a smooth vector field on D. Then

$$\int_C P\, dx + Q\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\, dA$$



Let C be a curve traversed once by $\vec{r}(t) = \langle x(t), y(t)\rangle$ for $a \le t \le b$.

C is closed if the initial and terminal points of C coincide, i.e. $\vec{r}(a) = \vec{r}(b)$.

C is simple if C has no self-intersections except at the endpoints, i.e. $\vec{r}(t_1) \neq \vec{r}(t_2)$ for all $a < t_1 < t_2 < b$.

C is positively oriented if it is traversed in the counterclockwise direction as t increases from a to b. The region D is always to the left of C.

Rmk. The line integral of a vector field along a curve depends on the orientation of the curve as follows: If $-C$ denotes the curve C traversed in the opposite direction, then $\int_{-C} f(x,y)\, dx = -\int_C f(x,y)\, dx$.

Ex. 5. Evaluate $\displaystyle\int_C (3y - e^{\sin x})\, dx + (7x + \sqrt{y^4+1})\, dy$ where C is the circle $x^2 + y^2 = 9$ traversed in the counterclockwise direction.

Ex. 6. Evaluate $\int_C y^2\, dx + 3xy\, dy$ where C is the boundary of the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ in the upper half plane, traversed counterclockwise.

Ex. 7. Let C be the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$, oriented counterclockwise and let $\vec{F}(x,y) = \langle x^3, xy \rangle$.

(a) According to Green's Theorem, the line integral $\int_C \vec{F} \cdot d\vec{r} = \int_C x^3 \, dx + xy \, dy$ is equal to a certain double integral. Set up and evaluate this double integral.

(b) Verify Green's Theorem by evaluating the line integral directly.

Ex. 8. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y) = \langle xy^2, x^2y \rangle$ and C is traced out by $\vec{r}(t) = \langle \cos(t), 2\sin(t) \rangle$ for $0 \le t \le \frac{\pi}{2}$.

Ex. 9. Consider the vector field $\vec{F}(x,y) = \langle e^{3x} + xy, e^{3y} - xy \rangle$. Evaluate the integral $\displaystyle\int_C \vec{F} \cdot d\vec{r}$ where C is the unit circle $x^2 + y^2 = 1$ oriented counterclockwise.

Ex. 10. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where C is the union of line segment C_1 from $(2,0,0)$ to $(3,4,5)$ followed by the vertical line segment C_2 from $(3,4,5)$ to $(3,4,0)$ for the following vector fields $\vec{F}$.

(a) $\vec{F}(x,y,z) = \langle y^2, 2xy, 4z \rangle$

(b) $\vec{F}(x,y,z) = \langle y, z, x \rangle$