Efficient Ex Post Implementable Auctions and English Auctions for Bidders with Non-Quasilinear Preferences

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Abstract

We study efficient auction design for a single indivisible object when bidders have interdependent values and non-quasilinear preferences. Instead of quasilinearity, we assume only that bidders have positive wealth effects. Our setting nests cases where bidders are ex ante asymmetric, face financial constraints, are risk averse, and/or face ensuing risk. We give necessary and sufficient conditions for the existence of an ex post implementable and (ex post Pareto) efficient mechanism. These conditions differ between the standard case where the auctioneer is a seller and when the auctioneer is a buyer (a procurement auction).

When the auctioneer is a seller, there is an efficient ex post implementable mechanism if there is an efficient ex post implementable mechanism in a corresponding quasilinear setting. This result extends established results on efficient ex post equilibria of English auctions with quasilinearity to our non-quasilinear setting. Yet, in the procurement setting there is no mechanism that has an efficient ex post equilibrium if the level of interdependence between bidders is sufficiently strong. This result holds even if bidder costs satisfy standard single crossing conditions that are sufficient for efficient ex post implementation in the quasilinear setting.

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1 Introduction

Efficient auction design is a central question in mechanism design. In the private value single unit quasilinear benchmark case, the English auction has an efficient dominant strategy equilibrium. More recent research gives necessary and sufficient conditions for the English auction to have an efficient ex post equilibrium when bidders have interdependent values. Thus, there are well-understood settings where the English auction's efficient equilibrium is robust to asymmetries across bidders' beliefs and higher order beliefs.

While these notable results on English auctions show that it is robust to asymmetries in bidder beliefs, these results require the strong assumption that bidder preferences are quasilinear. In many auction settings, bidders do not have quasilinear preferences, and violations of quasilinearity are frequently reported. For example, Maskin (2000) argues that financial market imperfections may result in liquidity-constrained bidders. Salant (1997) draws on his personal consulting experience to argue that financial constraints are a salient feature of how bidders determine their bids. In addition to access to credit, risk aversion and wealth effects are important features of auctions for larger items like houses.[1]

In this paper, we study the efficient auction design problem for a single indivisible unit when bidders have interdependent values. We remove the quasilinearity restriction on bidder preferences, and assume only that their preferences exhibit positive wealth effects. Our setting is sufficiently general to allow for asymmetric bidders who are risk averse, have financing constraints, have budgets, or face ensuing risk. Like much of the related literature on efficient auction design, we study auctions that are ex post implementable. Thus, our predictions are robust to asymmetries in bidders beliefs and higher order beliefs. Our main contribution is to provide conditions under which existing results from the literature on efficient auction design with quasilinearity can be extended to study efficient design on a more general preference domain. Interestingly, we show that the necessary and sufficient conditions for the existence of an efficient and ex post implementable auction differ depending on whether the auctioneer is a buyer or a seller.

Removing quasilinearity complicates the efficient auction design problem. With quasilinearity, an auction outcome is Pareto efficient if and only if the bidder with the highest value (or lowest cost) wins. Hence, the space of efficient allocations is independent of bidder transfers. Yet, in our non-quasilinear setting, the presence of wealth effects imply that a bidder's demand for a unit following the auction depends on the amount they paid (or were paid) in the auction. With wealth effects it is possible that the bidder with the highest willingness to pay wins the auction and that there are Pareto improving trades between bidders

[1] Homes are often sold via auction. For example, in Melbourne, Australia an estimated 25-50% of homes are sold via auction (see Mayer (1998)).

following the auction. Thus, the space of (ex post Pareto) efficient outcomes depends on both the allocation of the object and bidder transfers. While the space of efficient outcomes is qualitatively different without quasilinearity, our first result (Theorem 1) shows that we are able to extend results describing efficient and ex post implementable auctions for bidders with quasilinear preferences to our non-quasilinear setting when the auctioneer is a seller. The theorem states that there is an auction with an efficient ex post equilibrium if there is an auction with an efficient ex post equilibrium in a corresponding quasilinear setting, in which each bidder's valuation is equal to her willingness to pay in the non-quasilinear setting.

The existence of an efficient and ex post implementable auction in a corresponding quasilinear setting is a sufficient condition for the existence of such a mechanism in our non-quasilinear setting because positive wealth effects amplify the efficiency of the mechanism relative to the corresponding quasilinear setting. With positive wealth effects the winning bidder feels wealthier when she wins the good and pays a price that is below her willingness to pay. The increase in the winner's perceived wealth increases her willingness to sell the good relative to her willingness to pay because we assume bidders have positive wealth effects. Thus, the winner is relatively less inclined to trade with her rivals. We use this observation to show that an auction outcome that is efficient in the corresponding quasilinear setting is also efficient in our non-quasilinear setting. A corollary of Theorem 1 is that the English auction has an efficient ex post equilibrium when the auctioneer is a seller if bidders' willingnesses to pay satisfy the crossing conditions established by Maskin (1992), Krishna (2003), or Birulin and Izmalkov (2011).

However, the implications of Theorem 1 do not extend to procurement settings. When bidders are competing to sell a good to the auctioneer, positive wealth effects make the winning bidder more inclined to trade with her rivals because she is paid an amount in the auction that exceeds her reservation cost of supplying a unit. This makes the winning bidder feel wealthier and the increase in the winner's wealth increases her demand for a unit relative to her losing rivals. The winning bidder is therefore more inclined to trade with her rivals following the auction. In fact, we show that there are cases where the English auction has an ex post equilibrium in which the bidder with the lowest reservation cost supplies the auctioneer with the good, but the auction outcome is inefficient. The outcome is inefficient because the auctioneer's payment to the winner increases her demand for repurchasing good to such an extent that an ex post Pareto improving trade is created between the auction winner and one of the losers. Theorem 2 formalizes this result for the case where bidders have strictly positive wealth effects. The theorem shows that there is no procurement auction that retains the English auction's desired incentive and efficiency properties — ex post implementability and Pareto efficiency — when the degree of interdependence in bidder preferences is sufficiently strong.

Related Literature

Prior work on efficient auction design without quasilinearity has primarily focused on private value settings. Much of this work studies either revenue maximizing auctions (see Maskin and Riley (1984), Laffont and Robert (1996), Pai and Vohra (2014), and Baisa (2017a)), or bid behavior in standard auctions when bidders have private values and non-quasilinear preferences (see, for example, Matthews (1983, 1987), Che and Gale (1996, 1998, 2006)).

More toward the focus of this paper, there is also a literature that studies efficient auction design when bidders have non-quasilinear preferences. Within this literature, there are a number of papers that study auction design when bidders have a particular violation of quasilinearity — a hard budget constraint. In particular, Maskin (2000); Dobzinski, Lavi, and Nisan (2012); and Pai and Vohra (2014) all study efficient auction design when bidders have hard budgets. There is also a literature that studies efficient auction design on a general preference domain, like the one studied in this paper. Saitoh and Serizawa (2008), Morimoto and Serizawa (2015), and Baisa (2017b) all provide necessary and sufficient conditions for efficient auction design in non-quasilinear settings where we do not make function form restrictions on bidder utility functions. In addition, like this paper, these three aforementioned papers study the design of auctions that are robust to asymmetries in bidder higher order beliefs and auctions that implement an ex post Pareto efficient allocation of resources.

Saitoh and Serizawa (2008) shows that the Vickrey rule is the unique mechanism that satisfies desirable incentive and efficiency properties when bidders have private values and non-quasilinear preferences. Our results show that Saitoh and Serizawa's positive implementation result does not extend to interdependent value settings, even if bidder demands satisfy single crossing conditions that ensure the existence of efficient ex post equilibria in quasilinear settings.

There are fewer papers that study auctions with the interdependent value and non-quasilinear preferences. The exceptions are Burkett (2015), Kotowski (2017), Fang and Parreiras (2002, 2003), and Hu, Matthews, and Zou (2015). Burkett studies bid behavior in first- and second price auctions when bidders have interdependent values and budgets. Kotowski studies first-price auctions in a similar model. The Fang and Parreiras papers study revenue implications of information disclosure in English auctions when bidders have budgets and interdependent values. The focus of Hu, Matthews, and Zou (2015) is closer to our paper. They study the efficiency properties of the English auctions when bidders are risk averse and face ensuing risk. In Section 5 we show that ensuing risk is nested as a particular case of our normal good setting.

Outside of the auctions literature, Nöldeke and Samuelson (2015) use an implementation duality to study principal agent problems without quasilinearity. We take a similar approach by using a mapping from a non-quasilinear setting to a quasilinear setting in order to study auction design.

The remainder of the paper proceeds as follows. After presenting a motivating example in Section 2, Section 3 introduces our formal model. Section 4 shows the relationship between ex post implementation without quasilinearity and ex post implementation in a corresponding quasilinear setting. Section 5 gives results on the setting in which the auctioneer is a seller, and Section 6 then studies the procurement setting without quasilinearity.

2 Motivating Examples

We start by presenting two motivating examples that illustrate the main insights of the paper. Both examples are of English auctions. The first example is a standard auction setting where bidders with positive wealth effects compete to buy a good; and the second example is a procurement setting where bidders with positive wealth effects compete to sell a good to the auctioneer. In both cases there are ex post equilibria in which the bidder with the highest willingness to pay (or lowest cost) wins the auction. However, it is only in the first case that the equilibrium outcome is ex post Pareto efficient. Wealth effects explain the difference in results between the two cases.

2.1 An English Auction for a Normal Good

Consider an English auction where the auctioneer sells a single indivisible good. There are two bidders. Bidder i has a private signal $s_i \in [0,1]$, initial wealth of 1, and log-utility for money. Her utility is

$$s_i + \alpha s_j + ln(1-p)$$

if she wins the good and pays p. Bidder i gets utility $ln(1-p)$ if she does not win the good and pays price p. Note that α measures the degree of interdependence in bidders' preferences. We assume that $\alpha \in [0,1]$. A simple calculation shows that bidder i is willing to pay $d(s_i, s_j) = 1 - e^{-(s_i+\alpha s_j)}$ to win the good if her signal is s_i and her rival's signal is s_j.[2]

The English auction has an ex post equilibrium where bidder i drops out when the price

[2]Where we define bidder i's willingness to pay for a unit $d(s_i, s_j)$ as solving the expression

$$s_i + \alpha s_j + ln(1 - d(s_i, s_j)) = ln(1) = 0 \implies 1 - d(s_i, s_j) = e^{-(s_i+\alpha s_j)} \implies d(s_i, s_j) = 1 - e^{-(s_i+\alpha s_j)}.$$

reaches $d(s_i, s_i)$, which is analogous to an ex post equilibrium in a quasilinear setting, such as the one in Milgrom and Weber (1982), in which the bidder drops out at her expected valuation conditional on her opponent receiving the same signal.

The ex post equilibrium of the English auction assigns the good to the bidder with the highest willingness to pay. However, it is not immediate that the outcome is Pareto efficient. With wealth effects, the winning bidder's willingness to sell depends on the price she paid to win. In this example the English auction is (ex post Pareto) efficient because there are no ex post Pareto improving trades among bidders. To see this, suppose that $s_1 > s_2$. In that case, bidder 1 wins the good and pays $d(s_2, s_2)$ to win. A straightforward calculation shows that bidder 1's willingness to sell the good conditional on winning and paying $d(s_2, s_2)$ is c where $c := e^{s_1 - s_2} - e^{-(1+\alpha)s_2}$.[3] A second calculation shows that bidder 1's willingness to sell, c, exceeds her willingness to pay for the good, $d(s_1, s_2)$.[4] Thus, the winning bidder's willingness to sell is larger than the amount she was willing to pay for the good, because the winning bidder has positive wealth effects.

If the winning bidder wins the good and pays exactly her willingness to pay, then her willingness to sell the good following the auction equals her willingness to pay. Yet, in the English auction, the winning bidder pays weakly less than her willingness to pay to win the good. Therefore, the auction outcome is as though the winning bidder wins the good, pays her willingness to pay for the good, and is then given a partial refund. The refund increases the bidder's willingness to sell the good relative to her initial willingness to pay, because she has positive wealth effects. Since the winning bidder has the highest willingness to pay before receiving the good, her willingness to sell after winning must exceed her rival's willingness to pay and there are no Pareto improving trades. This logic suggests a close connection between ex post efficiency results in quasilinear settings and ex post Pareto efficiency in a non-quasilinear setting in which bidders have positive wealth effects. Theorem 1 generalizes this intuition.

If bidders have negative wealth effects or the auctioneer is a buyer, as in procurement, the results change. In the procurement case, there exist ex post equilibria in which the winning supplier has the lowest cost but the outcome is Pareto inefficient. The result is suggested by reversing some of the logic in the previous paragraph as we argue next.

[3]Bidder 1 gets utility $s_1 + \alpha s_2 + ln(1 - d(s_2, s_2))$ when she wins. The bidder's willingness to sell is given by $c = e^{s_1 - s_2} - e^{-(1+\alpha)s_2}$, where c solves

$$s_1 + \alpha s_2 + ln(1 - d(s_2, s_2)) = ln(1 - d(s_2, s_2) + c).$$

We derive the value of c from the above equation in the appendix.

[4]See the appendix for the calculation.

2.2 An English Auction in a Procurement Setting with Positive Wealth Effects

Consider a similar setting where two bidders compete to supply a good in an English procurement auction. Each bidder begins with a unit of the good. Bidder i has a private signal $s_i \in [0,1]$, initial wealth of 1, and log-utility for money. Her utility is

$$ln(1+p)$$

if she supplies her unit of the good to the auctioneer and is paid p. She gets utility

$$(s_i + \alpha s_j) + ln(1+p)$$

if does not supply the good, and is paid p. Bidder i's reservation cost is $c(s_i, s_j) = e^{(s_i+\alpha s_j)} - 1$ if her signal is s_i and her rival's signal is s_j.[5]

The English auction has an ex post equilibrium in this setting. Bidder i drops out when the price falls to $c(s_i, s_i)$ and thus the bidder with the lowest reservation cost always supplies the auctioneer with her unit. While the English auction has an ex post equilibrium where the lowest cost bidder always wins, the auction does not necessarily satisfy ex post Pareto efficiency because the presence of wealth effects changes the winning bidder's incentive to trade with her rival after being paid by the auctioneer. To see this, suppose that $s_1 < s_2$. Then, bidder 1 has the lower reservation cost, wins the procurement auction, and is paid $c(s_2, s_2)$ to supply her unit of the good. After the auction is over, bidder 1 is willing to pay bidder 2 the amount $d = e^{(1+\alpha)s_2} - e^{s_2 - s_1}$ to buy her unit.[6] There is an ex post Pareto improving trade between the two bidders if bidder 1's willingness to pay for a unit exceeds bidder 2's reservation cost of supplying a unit. This occurs if

$$d = e^{(1+\alpha)s_2} - e^{s_2-s_1} > e^{s_2+\alpha s_1} - 1 = c(s_2, s_1).$$

The above condition holds for any fixed $s_2 > s_1$ when α is sufficiently close to one.[7] Thus, the auction outcome is such that there are ex post Pareto improving trades where the winning bidder buys a unit from her rival bidder if the degree of interdependence is sufficiently strong.

[5] Formally, we define $c(s_i, s_j)$ as solving

$$ln(1 + c(s_i, s_j)) = (s_i + \alpha s_j) + ln(1) = s_i + \alpha s_j \implies c(s_i, s_j) = e^{(s_i+\alpha s_j)} - 1.$$

[6] Where d is the amount that makes bidder 1 indifferent between (1) receiving transfer $c(s_2, s_2)$ and no unit and (2) receiving transfer $c(s_2, s_2) - d$ and receiving a unit. See appendix for calculation.

[7] This is shown in the appendix.

Bidder 1 pays bidder 2 a price $p \in (e^{s_2+\alpha s_1} - 1, e^{(1+\alpha)s_2} - e^{s_2-s_1})$ in the Pareto improving trade.

The inefficiency of the English auction is caused by the presence of wealth effects. This is most easily seen in the pure common value case where $\alpha = 1$. In that case, the above inequality holds for all $s_2 > s_1$. Thus, every outcome of the English auction is a Pareto dominated outcome. Note that in the pure common value case both bidders have the same reservation cost and the winning bidder is the bidder with the lower signal. The winning bidder supplies her unit and is paid an amount that exceeds her reservation cost. This makes the winning bidder feel wealthier and increases her demand for a unit. Thus, the winning bidder is willing to pay some amount that is greater than her reservation cost for her rival's unit. Therefore, there is a Pareto improving trade between the winning bidder and her losing rival, because the losing bidder's reservation cost for supplying a unit is unchanged, and the winning bidder's willingness to pay for a unit exceeds the common reservation cost of supplying a unit. The example shows that the presence of positive wealth effects in procurement settings makes the winning bidder more inclined to trade with her rival following the auction outcome. Theorem 2 generalizes the intuition described in the above example to a general procurement setting where bidders have strictly positive wealth effects. Specifically, we show that when there is sufficiently strong interdependence between bidders' preferences, there is no auction that is individual rational, ex post incentive compatible, ex post Pareto efficient, and does not use subsidies.

3 Model

3.1 Two Settings

Bidders $1, \ldots, n$ have unit demand for an indivisible good. We study efficient auction design in two settings. The first setting we study is the canonical auction setting where the auctioneer is endowed with a single indivisible good and each bidder starts without a unit of the good. In this setting the auctioneer is a seller, and we refer to it as the sell setting, using the label $\mathbb{S}$ for it. In the sell setting, we assume that the auctioneer is risk neutral and has zero value for the good.

The second setting is a procurement auction where each bidder is endowed with a single unit of the good and the auctioneer begins with no units. We refer to this setting as the procurement setting and use the label $\mathbb{P}$ for it. In the procurement setting, we assume that the auctioneer is risk neutral and has perfectly inelastic demand for a single unit (i.e. unit demand with infinite value for the unit).

We let $\omega^c \in \{0,1\}$ be the number of units that a bidder $i \in \{1,\ldots,n\}$ is endowed with in setting $c \in \{\mathbb{S},\mathbb{P}\}$. Thus, we have assumed $\omega^{\mathbb{S}} = 0\ \forall i \in \{1,\ldots,n\}$ in the sell setting $\mathbb{S}$ and $\omega^{\mathbb{P}} = 1\ \forall i \in \{1,\ldots,n\}$ in procurement setting $\mathbb{P}$.

3.2 Bidder Preferences

Bidder preferences are modeled in the same way in both settings. Bidder $i \in \{1,\ldots,n\}$ receives a private signal $s_i \in S_i \subset \mathbb{R}^k$.[8] Define $S^n := \times_{i=1}^n S_i$ and $S^{-i} := \times_{j\neq i} S_j$. We assume bidders are utility maximizing and bidder i's preferences are described by utility functions u_i^x where

$$u_i^x : \mathbb{R} \times S^n \to \mathbb{R},$$

and $x \in \{0,1\}$ indicates whether the bidder owns a unit of the good or not. Bidder i has unit demand and gets utility $u_i^x(t,s)$ when she wins $x \in \{0,1\}$ units, receives the transfer $t \in \mathbb{R}$, and all bidders' signals are given by $s := (s_1,\ldots,s_n) \in S^n$.[9] We let $u := (u_1,\ldots,u_n)$ where $u_i := (u_i^0, u_i^1)$. We assume that $u_i^x(t,s)$ is continuous in (t,s) and strictly increasing and differentiable in t for all $x \in \{0,1\}$ and $s \in S^n$.

A bidder's utility increases from owning a good, holding all else equal,

$$u_i^1(t,s) > u_i^0(t,s)\ \forall i \in \{1,\ldots,n\},\ t \in \mathbb{R},\ s \in S^n.$$

We assume that bidders have finite willingnesses to pay and willingnesses to sell. Thus, for any bidder i and signal $s \in S^n$ and $t \in \mathbb{R}$, there exist a $p > 0$ and a $p' > 0$ such that $u_i^0(t,s) > u_i^1(t-p,s)$ and $u_i^0(t+p',s) > u_i^1(t,s)$. We say that bidder preferences are **classical** if $u = (u_1,\ldots,u_n)$ satisfies the aforementioned assumptions.[10]

Definition 1. (Classical preferences)

Bidders have classical preferences if $u = (u_1,\ldots,u_n)$ is such that:

1. A bidder gets positive utility from owning a unit: $u_i^1(t,s) > u_i^0(t,s)\ \forall i \in \{1,\ldots,n\},\ t \in \mathbb{R},\ s \in S^n$.

[8] Some of our results are for the case where a bidder's signal space is one-dimensional $S_i \subset \mathbb{R}$. Indeed, there are many impossibility results already in the quasilinear setting on efficient implementation with multi-dimensional types (see Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001), Jehiel et al. (2006)). This is discussed in greater detail at the end of Section 4.

[9] Note, that it is without loss of generality to assume that a bidder begins with her initial wealth normalized to zero. If a bidder has preferences $\hat{u}_i$ and initial wealth $w_i \in \mathbb{R}$, then we can define u_i has being $u_i^x(t,s) = \hat{u}_i^x(t+w_i,s)\ \forall x \in \{0,1\},\ t \in \mathbb{R},\ s \in S^n$. Alternatively, a bidder's initial wealth could be a dimension of her private information.

[10] The term classical preferences follows from the literature on auctions with non-quasilinear preferences. See Saitoh and Serizawa (2008) and Morimoto and Serizawa (2015).

2. Utility is continuously increasing in money: $u_i^x(\cdot, s)$ is continuous, differentiable, and strictly increasing $\forall\, i \in \{1, \ldots, n\},\ x \in \{0, 1\},$ and $s \in S^n$.

3. The willingness to pay and sell are finite: for any $i \in \{1, \ldots, n\}$, $t \in \mathbb{R}$, and $s \in S^n$, there exist $p, p' \in (0, \infty)$ such that $u_i^0(t, s) > u_i^1(t - p, s)$ and $u_i^0(t + p', s) > u_i^1(t, s)$.

Our main results consider the case where bidder preferences exhibit positive wealth effects. Intuitively, positive wealth effects imply that a bidder's demand for a unit of the good increases when we increase the bidder's initial wealth.

Definition 2. (Positive wealth effects)

Consider a setting in which bidders have classical preferences. We say that bidders have weakly positive wealth effects if u is such that

$$u_i^1(t - p, s) \geq u_i^0(t, s) \implies u_i^1(t' - p, s) \geq u_i^0(t', s),\ \forall p > 0, t' > t,\ i \in \{1, \ldots, n\},\ s \in S^n,$$

and strictly positive wealth effects if u is such that

$$u_i^1(t - p, s) \geq u_i^0(t, s) \implies u_i^1(t' - p, s) > u_i^0(t', s),\ \forall p > 0, t' > t,\ i \in \{1, \ldots, n\},\ s \in S^n.$$

The definition of weakly (strictly) positive wealth effects implies that a bidder's demand for a unit of the good is weakly (strictly) increasing as she becomes wealthier. Note that quasilinear preferences satisfy weakly positive wealth effects, but not strictly positive wealth effects.

It is useful to find transfers that make bidder i indifferent between receiving and not receiving the good. We define the functions $d_i^1(t, s)$ and $d_i^0(t, s)$ implicitly as

$$u_i^1(t - d_i^1(t, s), s) = u_i^0(t, s),$$

$$u_i^1(t, s) = u_i^0(t + d_i^0(t, s), s).$$

Thus, a bidder who does not own a unit and has received transfers $t \in \mathbb{R}$ would be indifferent between remaining at her status quo and buying a unit for price $d_i^1(t, s)$. We call this bidder i's willingness to pay for the good. Similarly, a bidder who owns a unit and has received transfers t would be indifferent between remaining at her status quo and selling her unit for price $d_i^0(t, s)$. We call this bidder i's reservation cost for supplying a unit of the good (i.e., her willingness to sell).

Remark 1. Suppose that bidders have classical preferences. The functions $d_i^0(t, s)$ and $d_i^1(t, s)$ have the following properties for all $i \in \{1, \ldots, n\}$, $t \in \mathbb{R}$ and $s \in S^n$:

1. $d_i^0(\cdot, s)$ and $d_i^1(\cdot, s)$ are continuous, differentiable and non-negative.

2. $d_i^0(t, s) = d_i^1(t + d_i^0(t, s), s)$ and $d_i^1(t, s) = d_i^0(t - d_i^1(t, s), s)$.

3. If bidders have weakly (strictly) positive wealth effects, $d_i^0(\cdot, s)$ and $d_i^1(\cdot, s)$ are weakly (strictly) increasing.

Note that the first point follows from continuity. The second point follows from the definitions of d_i^0 and d_i^1. The final point follows from Definition 2.

3.3 Mechanisms

By the revelation principle, it is without loss of generality to restrict attention to direct revelation mechanisms. In addition, we restrict attention to mechanisms with deterministic assignment rules.[11]

A direct revelation mechanism is a mapping that specifies (1) bidder transfers and (2) a feasible assignment of the good, for any profile of bidder signals $s \in S^n$. We use a $c \in \{\mathbb{S}, \mathbb{P}\}$ superscript to denote whether we are in the sell setting or the procurement setting. Thus, a direct revelation mechanism Γ^c in setting $c \in \{\mathbb{S}, \mathbb{P}\}$ consists of a transfer rule $t : S^n \to \mathbb{R}^n$ and an allocation rule $q : S^n \to Y^c$, where we let $Y^c \subset \{0, 1\}^n$ be the space of feasible assignments.

In the sell setting $\mathbb{S}$, the space of feasible assignments is $Y^{\mathbb{S}} := \{y \in \{0, 1\}^n | \sum_{i=1}^n y_i = 1\}$. In words, a feasible assignment specifies a single winning bidder of the good. Note that we do not consider mechanisms where the good is not assigned to any bidder.[12] We similarly define the space of feasible assignments in the procurement setting to be $Y^{\mathbb{P}} := \{y \in \{0, 1\}^n | \sum_{i=1}^n y_i = n - 1\}$. In words, a feasible assignment specifies exactly one bidder who supplies the auctioneer with a unit.

[11]We limit attention to auctions that have deterministic outcomes. This assumption is partly motivated by practical concerns — randomization is rarely used in practice. Moreover, from a theoretical perspective, prior research shows that there is no mechanism that has desirable incentive and efficiency properties when we allow for randomization. When we allow for stochastic mechanisms, efficient auction design for a single unit when bidders are non-quasilinear is equivalent to efficient auction design in a multi-unit setting. Bidders have downward sloping demand for additional probability units (see Baisa (2017a)). Moreover, Baisa (2017b) shows that efficient auction design in a multi-unit setting is generally impossible even with private values. Hence, the restriction to deterministic mechanisms is a natural second-best analysis. Furthermore, related literature on efficient auctions without quasilinearity similarly restricts attention to deterministic allocation rules (see Saitoh and Serizawa (2008), Morimoto and Serizawa (2015), and Hu, Matthews, and Zou (2015)).

[12]It is without loss of generality to restrict attention to assignments in which the seller always sells the good in the sell setting or buys the good in the procurement setting. We study auctions that implement efficient assignments, and under our assumptions it is common knowledge that trade with the seller takes place in any ex post efficient assignment in either setting.

We study four different properties that a mechanism Γ^c can satisfy: (1) ex post incentive compatibility, (2) individual rationality, (3) no subsidies, and (4) ex post Pareto efficiency. We define each property below.

A mechanism Γ^c satisfies ex post incentive compatibility if truthful reporting is always a Nash equilibrium of the game in which the signal realization $(s_1, \ldots, s_n) \in S^n$ is common knowledge.

Definition 3. (Ex post incentive compatibility)

Fix $c \in \{\mathbb{S}, \mathbb{P}\}$. A mechanism Γ^c is ex post incentive compatible (EPIC) if for all $i \in \{1, \ldots, n\}$, $s_i, s_i' \in S_i$, and $s_{-i} \in S^{-i}$,

$$u_i^{q_i(s_i,s_{-i})}(t_i(s_i, s_{-i}), (s_i, s_{-i})) \geq u_i^{q_i(s_i',s_{-i})}(t_i(s_i', s_{-i}), (s_i, s_{-i})).$$

A mechanism satisfies ex post individual rationality if a bidder is made no worse off by participating for every realization of signals. Recall ω^c is a bidder's endowment of units (i.e., $\omega^{\mathbb{P}} = 1$ and $\omega^{\mathbb{S}} = 0$).

Definition 4. (Ex post individual rationality)

Fix $c \in \{\mathbb{S}, \mathbb{P}\}$. A mechanism Γ^c is ex post individually rational (IR) if for all $s \in S^n$ and $i \in \{1, \ldots, n\}$,

$$u_i^{q_i(s)}(t_i(s), s) \geq u_i^{\omega^c}(0, s).$$

We say that a mechanism satisfies no subsidies if losing bidders do not receive positive transfers.

Definition 5. (No subsidies)

Fix $c \in \{\mathbb{S}, \mathbb{P}\}$. A mechanism Γ^c satisfies no subsidies if a losing bidder is never paid to participate. That is,

$$q_i(s) = \omega^c \implies t_i(s) \leq 0 \ \forall s \in S^n, \ i \in \{1, \ldots, n\}.$$

The space of efficient outcomes depends on both the assignment and the transfers. Like much of the literature on efficient auction design without quasilinearity, we consider mechanisms that implement ex post Pareto efficient outcomes as defined by Holmstrom and Myerson (1983). An outcome is ex post Pareto efficient if there are no ex post Pareto improving trades among bidders. This is the same efficiency notion that Saitoh and Serizawa (2008); Dobzinski, Lavi, and Nisan (2012); Hu, Matthews, and Zou (2015); and Morimoto and Serizawa (2015) use to study efficient auction design without quasilinearity.

Definition 6. (Ex post Pareto efficient)

Fix $c \in \{\mathbb{S}, \mathbb{P}\}$. Mechanism Γ^c is ex post Pareto efficient if for any $s \in S^n$,

$$d_i^0(t_i(s), s) \geq d_j^1(t_j(s), s)$$

for all $i \in \{1, \ldots, n\}$ such that $q_i(s) = 1$ and $j \in \{1, \ldots, n\}$ such that $q_j(s) = 0$.

In words, a mechanism is ex post Pareto efficient if for any type profile $s \in S^n$, the outcome is such that the willingness to sell of any bidder who finishes with a unit exceeds the willingness to pay of any bidder who does not finish with a unit. Thus, there are no ex post Pareto improving trades between bidders for all type realizations $s \in S^n$. Without quasilinearity, requiring ex post Pareto efficiency is not equivalent to requiring that the auction assign the object to the bidder with the highest willingness to pay. Later in the paper, we show that assigning the good to the bidder with the highest willingness to pay is neither necessary nor sufficient for an auction to be ex post Pareto efficient (see Example 3 and Theorem 2 on procurement auctions).

4 A Corresponding Quasilinear Setting

In this section, we define a corresponding quasilinear setting for a mechanism Γ^c when bidders have classical preferences $u = (u_1, \ldots, u_n)$. Bidders in the corresponding quasilinear setting have quasilinear preferences and valuations for units that induce the same ordinal preference ranking over the outcomes of a mechanism Γ^c as the bidders in the non-quasilinear setting. We show that by studying the mechanism in a specific quasilinear setting, we can apply some previously established results on efficient ex post implementation with quasilinearity to our non-quasilinear setting.

Our definition of the quasilinear setting is simplified by appealing to the taxation principle (Rochet (1985)). The taxation principle suggests that a mechanism is ex post implementable only if each bidder's payment is constant in her reported type given the number of units she wins.

Lemma 1. *Fix $c \in \{\mathbb{S}, \mathbb{P}\}$ and suppose that bidders have classical preferences. If mechanism Γ^c satisfies EPIC, then*

$$q_i(s_i, s_{-i}) = q_i(s_i', s_{-i}) \implies t_i(s_i, s_{-i}) = t_i(s_i', s_{-i}) \; \forall s_i, s_i' \in S_i, \; s_{-i} \in S^{-i}, \; i \in \{1, \ldots, n\}.$$

A Corollary of Lemma 1 is that any mechanism that satisfies EPIC has an allocation-contingent transfer rule $\tau_i^x(s_{-i})$ which states the amount that bidder i receives in transfers in mechanism Γ^c when she is allocated $x \in \{0,1\}$ units and her rivals report types $s_{-i} \in S^{-i}$.

Corollary 1. *Fix $c \in \{\mathbb{S}, \mathbb{P}\}$ and suppose that bidders have classical preferences. If mechanism Γ^c satisfies EPIC, then there exist functions $\tau_i^0, \tau_i^1 : S^{-i} \to \mathbb{R}$ such that*

$$\tau_i^0(s_{-i}) = \begin{cases} t_i(s_i, s_{-i}) & \text{if } \exists s_i \in S_i \text{ s.t. } q_i(s_i, s_{-i}) = 0, \\ 0 & \text{if } q_i(s_i, s_{-i}) \neq 0 \ \forall s_i \in S_i, \end{cases} \tag{1}$$

and

$$\tau_i^1(s_{-i}) = \begin{cases} t_i(s_i, s_{-i}) & \text{if } \exists s_i \in S_i \text{ s.t. } q_i(s_i, s_{-i}) = 1, \\ 0 & \text{if } q_i(s_i, s_{-i}) \neq 1 \ \forall s_i \in S_i, \end{cases} \tag{2}$$

for all $i \in \{1, \dots, n\}$.

By Corollary 1, we may completely describe the transfer rule, t, of a mechanism Γ^c that is EPIC using functions τ_i^0 and τ_i^1 for each bidder i. We use the notation $\Gamma^c = \{q, \tau\}$ to describe a mechanism Γ^c where $\tau_i^0(\cdot)$ and $\tau_i^1(\cdot)$ are described by Equations (1) and (2). Note that we set $\tau_i^0(s_{-i}) = 0$ if it is ever the case that $s_{-i} \in S^{-i}$ is such that $q_i(s_i, s_{-i}) = 1$ for all $s_i \in S_i$, meaning the mechanism allocates a unit to bidder i regardless of her report. The value of $\tau_i^0(s_{-i})$ does not influence bidder i's incentives in such cases, because bidder i does not win zero units in the mechanism and $\tau_i^0(s_{-i})$ is bidder i's payment conditional on winning zero units. Setting $\tau_i^0(s_{-i}) = 0$ in these cases simplifies our definition of the corresponding quasilinear setting (see Footnote 14). Similarly, we set $\tau_i^1(s_{-i}) = 0$ if $s_{-i} \in S^{-i}$ is such that $q_i(s_i, s_{-i}) = 0$, $\forall s_i \in S^{-i}$.

We define a corresponding quasilinear setting for a mechanism $\Gamma^c = \{q, \tau\}$ below.[13]

Definition 7. (Corresponding quasilinear setting)

Fix $c \in \{\mathbb{S}, \mathbb{P}\}$ and suppose that bidders have classical preferences. Associated with the mechanism $\Gamma^c = \{q, \tau\}$ is the corresponding quasilinear setting $QL(\Gamma^c, u)$ where bidder $i \in \{1, \dots, n\}$ has quasilinear preferences that are represented by the utility functions $u^x_{ql,i} : \mathbb{R} \times S^n \to \mathbb{R}$ where

$$u^x_{ql,i}(t, s) = v_i(s)x + t$$

[13] Note that Corollary 1 is not an if and only if statement. It says that EPIC mechanisms are a subset of the mechanisms that can be written in the form $\Gamma^c = \{q, \tau\}$. Thus, there are mechanisms that can be written in the form $\Gamma^c = \{q, \tau\}$ that are not EPIC. Furthermore, there are mechanisms that do not satisfy EPIC and cannot be written in the form $\Gamma^c = \{q, \tau\}$. We do not define a corresponding quasilinear setting for such mechanisms.

for $x \in \{0,1\}$. The valuation, $v_i : S^n \to \mathbb{R}$, is defined as

$$v_i(s) = d_i^{1-\omega^c}(\tau_i^{\omega^c}(s_{-i}), s).$$

Thus, we construct the corresponding quasilinear setting to be such that a bidder i with classical preferences described by the utility function u_i has the same ordinal preference ranking over the (deterministic) outcomes induced by mechanism Γ^c as a bidder with quasilinear preferences determined by the valuation $v_i(s) = d_i^{1-\omega^c}(\tau_i^{\omega^c}(s_{-i}), s)\ \forall s \in S^n$.[14]

Remark 2. Fix $c \in \{\mathbb{S}, \mathbb{P}\}$ and suppose that bidders have classical preferences. If mechanism $\Gamma^c = \{q, \tau\}$ satisfies IR and no subsidies in the corresponding quasilinear setting $QL(\Gamma^c, u)$ then

$$\tau_i^{\omega^c}(s_{-i}) = 0\ \forall s_{-i} \in S^{-i},\ i \in \{1, \ldots, n\}$$

and

$$v_i(s) = d_i^{1-\omega^c}(0, s)\ \forall s \in S^n,\ i \in \{1, \ldots, n\}.$$

Thus, if mechanism $\Gamma^c = \{q, \tau\}$ satisfies IR and no subsidy, a bidder i's value $v_i(s)$ in the corresponding quasilinear setting $QL(\Gamma^c, u)$ equals her willingness to pay having received no transfer, $d_i^1(0, s)$, in the sell setting. Similarly, her value equals her willingness to sell having received no transfer in the procurement setting.

Remark 3. Fix $c \in \{\mathbb{S}, \mathbb{P}\}$ and suppose that bidders have classical preferences.

Mechanism $\Gamma^c = \{q, \tau\}$ is EPIC in the setting $QL(\Gamma^c, u)$ if for all $s \in S^n,\ i \in \{1, \ldots, n\},\ s_i' \in S_i$,

$$v_i(s)q_i(s) + \tau_i^{q_i(s)}(s_{-i}) \geq v_i(s)q_i(s_i', s_{-i}) + \tau_i^{q_i(s_i', s_{-i})}(s_{-i}).$$

Mechanism Γ^c is IR in the setting $QL(\Gamma^c, u)$ if for all $s \in S^n,\ i \in \{1, \ldots, n\}$,

$$v_i(s)q_i(s) + \tau_i^{q_i(s)}(s) \geq \omega^c v_i(s).$$

Mechanism Γ^c is efficient in the setting $QL(\Gamma^c, u)$ if for all $s \in S^n,\ i \in \{1, \ldots, n\}$,

$$(q_1(s), \ldots, q_n(s)) \in \arg\max_{q \in Y^c} \sum_{i=1}^{n} v_i(s)q_i.$$

[14] Note that if for bidder i there is an $s_{-i} \in S^{-i}$ such that $q_i(s_i, s_{-i}) = 1 - \omega^c$ for all $s_i \in S_i$, the valuation is $v_i(s) = d_i^{1-\omega^c}(\tau_i^{\omega^c}(s_{-i}), s)$. As we note following Corollary 1, the value of $\tau_i^{\omega^c}(s_{-i})$ is irrelevant to bidder i in such a case, because this bidder receives the transfer $\tau_i^{1-\omega^c}(s_{-i})$ following any reported signal. We set $\tau_i^{\omega^c}(s_{-i}) = 0$ in such cases to make the bidder's valuation $v_i(s) = d_i^{1-\omega^c}(0, s)$ in the quasilinear setting, because it is this valuation that makes individual rationality in the corresponding quasilinear setting align with individual rationality in the non-quasilinear one (see the proof of Lemma 3).

Lemma 2 shows that a necessary and sufficient condition for Γ^c to satisfy EPIC is that Γ^c satisfy EPIC in the setting $QL(\Gamma^c, u)$.

Lemma 2. *Fix $c \in \{\mathbb{S}, \mathbb{P}\}$ and suppose that bidders have classical preferences. Mechanism $\Gamma^c = \{q, \tau\}$ satisfies EPIC if and only if mechanism Γ^c satisfies EPIC in the corresponding quasilinear setting $QL(\Gamma^c, u)$.*

The intuition for the proof of Lemma 2 follows directly from our construction of the corresponding quasilinear setting. We construct a mechanism's corresponding quasilinear setting to be such that bidders in the corresponding quasilinear setting have the same preference ranking over the menu of alternatives induced by the mechanism.

Lemma 3 establishes a similar result that allows us to verify IR by considering a mechanism's corresponding quasilinear setting.

Lemma 3. *Fix $c \in \{\mathbb{S}, \mathbb{P}\}$ and suppose that bidders have classical preferences and the mechanism $\Gamma^c = \{q, \tau\}$ satisfies EPIC. Mechanism Γ^c satisfies IR if and only if mechanism Γ^c satisfies IR in the corresponding quasilinear setting $QL(\Gamma^c, u)$.*

The proof of Lemma 3 similarly follows from the construction of our corresponding quasilinear setting. A non-quasilinear bidder and her corresponding quasilinear bidder have the same pairwise preference ranking between mechanism Γ^c's outcome and their status quo, as long as Γ^c is EPIC in the non-quasilinear setting. In addition, note that IR in the corresponding quasilinear setting implies IR in our non-quasilinear setting (and vice versa) in cases where $s_{-i} \in S^{-i}$ is such that $q_i(s_i, s_{-i}) = 1 - \omega^c \; \forall s_i \in S_i$, because we define the corresponding quasilinear setting to be such that $v_i(s) = d_i^{1-\omega^c}(0, s)$ in such cases.

We often study cases where a bidder's signal space is one-dimensional, $S_i \subset \mathbb{R}$. Indeed, there are many impossibility results in the quasilinear setting for implementation problems with multi-dimensional types (see Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001), and Jehiel et al. (2006)). In addition, all of our examples also impose that a bidder's willingness to pay for the object is monotone in her signal. Yet our general results do not require that we put structure on the signal space. Instead, our main results will show that we can study a non-quasilinear setting by considering a corresponding quasilinear setting. For example, if we study a setting where the signal space is multi-dimensional and there does not exist a non-trivial social choice functions that is EPIC in the mechanism's corresponding quasilinear setting, then Lemma 2 similarly shows that there is no non-trivial social choice function that is EPIC in our non-quasilinear setting.

Our contribution is to show when the existing tools on efficient auction design from the quasilinear setting can be used in our non-quasilinear setting. This problem is simplified by considering ex post equilibrium, because we are able to restrict attention to settings where bidders have complete information over the rivals' private information. However, the efficient auction design problem is complicated by the presence of wealth effects. In what follows, we show that the results from the literature on efficient auction design with quasilinear bidders can be extended to our non-quasilinear setting in the sell setting, when the object being sold is a normal good (Theorem 1). However, this positive result does not extend to the procurement setting where bidders compete to supply a good.

5 Auctions that sell a normal good

In this section we study efficient and EPIC mechanisms in the canonical sell setting $\mathbb{S}$ where bidders start without a good and compete to win a good. We assume that bidders have weakly positive wealth effects (i.e., the good is normal). We show that we can verify whether a mechanism satisfies EPIC, IR, and (ex post Pareto) efficiency in our non-quasilinear setting by studying whether there is a mechanism that satisfies the same properties in the corresponding quasilinear setting. In particular, the first main result shows that a mechanism is EPIC, IR, and efficient if the mechanism is EPIC, IR, and efficient in the corresponding quasilinear setting (Theorem 1). A useful corollary of this theorem is that we can use recent results in the literature on efficient ex post equilibria of English auctions in the quasilinear setting to obtain analogous sufficient conditions for the existence of efficient ex post equilibria of English auctions in the non-quasilinear setting (Corollary 2). For example, we show that an English auction has an efficient ex post equilibrium if bidders' willingness to pay functions $d_i^1(0, s)$ satisfy conditions given by Krishna (2003) or Birulin and Izmalkov (2011).

In the motivating example given in Section 2.1, bidders have additively separable preferences for the good and wealth. It is analogous to the financially constrained bidder case in Che and Gale (1998).[15] Positive wealth effects may also arise in situations where there is some residual uncertainty about the value of the good and bidders are risk averse, which the following example illustrates.

[15]In Che and Gale (1998), bidders have a (private) valuation for the good and are risk neutral, but bidders also borrow money to finance their auction bids, and the cost of borrowing is increasing. A special case of the financial constrained bidders case is the case where bidders have hard budgets. In this case, we assume that a bidder has a hard budget that is less than her valuation. This case is studied by Che and Gale (1996, 2000), Maskin (2000), and Pai and Vohra (2014), among others. In the hard budget case f_i is such that

$$f_i(t) = \begin{cases} t & \text{if } t \geq -b_i, \\ -\infty & \text{if } t < -b_i, \end{cases}$$

and b_i is bidder i's hard budget.

Example 1. (Ensuing risk) Bidder i maximizes her expected utility from money and she has decreasing absolute risk aversion. If bidder i wins the good, then she is paid a dividend of $g_i(s) + z_i$. The term z_i is a type-independent random variable. This case is studied by Hu, Matthews, and Zou (2015). The risky asset is a normal good because declining absolute risk aversion implies that a bidder becomes less risk averse as her wealth increases. Thus, she is willing to pay more for the risky asset when her wealth increases.

While the space of efficient outcomes in the quasilinear setting is distinct from the space of efficient outcomes in setting where bidders have weakly positive wealth effects, Theorem 1 shows that efficient auction design in the quasilinear setting is related to efficient design in the normal good setting.

Theorem 1. *Suppose that bidders have classical preferences that satisfy weakly positive wealth effects. A mechanism in the sell setting, $\Gamma^{\mathbb{S}} = \{q, \tau\}$, satisfies (1) EPIC, (2) IR and (3) efficiency in the non-quasilinear setting if $\Gamma^{\mathbb{S}}$ satisfies EPIC, IR, and efficiency in the corresponding quasilinear setting $QL(\Gamma^{\mathbb{S}}, u)$.*

Before discussing the intuition behind the proof, note that lemmas 2 and 3 show that if $\Gamma^{\mathbb{S}}$ satisfies (1) EPIC and (2) IR in $QL(\Gamma^{\mathbb{S}}, u)$, then $\Gamma^{\mathbb{S}}$ satisfies (1) and (2) in our non-quasilinear setting. Thus, the proof of Theorem 1 shows that if $\Gamma^{\mathbb{S}}$ satisfies (1) EPIC, (2) IR, and (3) efficiency in setting $QL(\Gamma^{\mathbb{S}}, u)$, then $\Gamma^{\mathbb{S}}$ satisfies efficiency in our non-quasilinear setting where bidders have weakly positive wealth effects.

The intuition for the result is the same as that given in our motivating example in Section 2.1. Consider a mechanism that satisfies the no subsidies condition, so that no losing bidder makes a transfer, $\tau_i^0(s_{-i}) = 0\ \forall s_{-i}$.[16] If the mechanism $\Gamma^{\mathbb{S}}$ is efficient in the corresponding quasilinear setting $QL(\Gamma^{\mathbb{S}}, u)$, then the mechanism assigns the good to the bidder with the highest willingness to pay. Thus the allocation rule is such that $q_i(s) = 1 \implies d_i^1(0, s) \geq d_j^1(0, s)\ \forall j \neq i$. If a bidder wins the good and pays her willingness to pay for the good, then her willingness to sell the good after the auction equals her willingness to pay, because $d_i^1(0, s) = d_i^0(-d_i^1(0, s), s)$. Thus after buying at price equal to her willingness to pay $d_i^1(0, s)$, the bidder is indifferent between selling and keeping the good at this price. Yet, the winning bidder pays a price, p, that is weakly below her willingness to pay. Since bidders have weakly positive wealth effects,

$$p \leq d_i^1(0, s) \implies d_i^1(0, s) = d_i^0(-d_i^1(0, s), s) \leq d_i^0(-p, s),$$

where the first equality holds by the construction of d_i^0 and d_i^1 and the final inequality holds because of weakly positive wealth effects. Therefore, the winning bidder's willingness to sell

[16]The formal proof for a general class of mechanisms is in the appendix, but the intuition is the same.

the unit after winning and paying p weakly exceeds her willingness to pay for the good. Moreover, the winner had the highest willingness to pay of all bidders. Thus, her willingness to sell after winning exceeds her rivals' willingnesses to pay, and hence, there are no ex post Pareto improving trades of the good.

Theorem 1 is a sufficient — but not necessary — condition to ensure a mechanism satisfies EPIC, IR, and efficiency. This is illustrated via Example 2 in Section 5.2. In the example, there are two ex ante symmetric bidders, and bidder 1 always wins the good for free. While bidder 1 does not necessarily have the highest willingness to pay, bidder 1's willingness to sell the good is sufficiently large (relative to her rival's willingness to pay for the good) because she wins the good for a low price (free). Thus, in our example, there are no ex post Pareto improving trades, even though the bidder with the highest willingness to pay may not win the good. We use a separate example to show that the implications of Theorem 1 do not hold when the assumption of weakly positive wealth effects is removed.

5.1 Application: English Auctions

Theorem 1 extends results on the efficiency of English auctions with quasilinearity to the non-quasilinear setting. Note that the English auction satisfies IR and the no subsidies condition by construction. Thus, we have that $\tau_i^0(s_{-i}) = 0 \ \forall s_{-i}$. Therefore, Theorem 1 states that the English auction has an efficient ex post equilibrium if there is an efficient ex post equilibrium of the corresponding quasilinear setting where bidder i has valuation $v_i(s) = d_i^1(0, s) \ \forall s \in S^n$.

Corollary 2. *Suppose that bidders have classical preferences that satisfy weakly positive wealth effects. The English auction has an efficient ex post equilibrium if the English auction has an efficient ex post equilibrium in the corresponding quasilinear setting where bidder i has valuation*

$$v_i(s) = d_i^1(0, s) \ \forall s \in S^n, \ i \in \{1, \dots, n\}.$$

Thus, we can say that an English auction has an efficient ex post equilibrium if $d_i^1(0, s)$ satisfies Krishna's (2003) average (or weighted) crossing conditions or the generalized single crossing condition of Birulin and Izmalkov (2011). Bidder strategies in the efficient ex post equilibrium in our setting with weakly positive wealth effects are identical to bidder strategies in the corresponding quasilinear setting, because Lemma 2 shows that an ex post equilibrium in the latter implies an equivalent ex post equilibrium in the former.

5.2 The Bounds of Theorem 1

Theorem 1 provides a sufficient condition for efficient auction design when bidder preferences have weakly positive wealth effects. The theorem shows that a mechanism $\Gamma^{\mathbb{S}}$ is EPIC, IR, and efficient in a setting with weakly positive wealth effects if the mechanism $\Gamma^{\mathbb{S}}$ is EPIC, IR, and efficient in the corresponding quasilinear setting $QL(\Gamma^{\mathbb{S}}, u)$. If mechanism $\Gamma^{\mathbb{S}}$ does not provide bidders with an upfront subsidy (i.e., $\tau_i^0(s_{-i}) = 0,\ \forall s_{-i} \in S^{-i},\ i \in \{1, \dots, n\}$) then Theorem 1 implies the mechanism $\Gamma^{\mathbb{S}}$ is efficient if the mechanism assigns the good to the bidder with the highest willingness to pay for the good. In this subsection, we show that assigning the good to the bidder with the highest willingness to pay is not a necessary condition for efficient auction design (Example 2).

In addition, we show that assigning the good to the bidder with the highest willingness to pay is not sufficient for efficient auction design if we remove the weakly positive wealth effects assumption and assume only that bidders have classical preferences. Thus, there are auctions with ex post equilibria in which the bidder with the highest willingness to pay is assigned the good but the allocation is inefficient (Example 3).

Example 2. (An efficient auction that does not assign the good to the bidder with the highest willingness to pay)

Suppose there are two bidders and that $S_1 = S_2 = [1, 2]$, where

$$u_i^x(t, s) = \left(s_i + \frac{1}{2}s_j\right)x + ln(t+1).$$

Then,

$$d_i^1(0, s) = 1 - e^{-(s_i + \frac{1}{2}s_j)},$$

and

$$d_i^0(t, s) = (1+t)(e^{s_i + \frac{1}{2}s_j} - 1).$$

See the appendix for derivations of d_i^0 and d_i^1. Let $\Gamma^{\mathbb{S}}$ be a mechanism where

$$q_1(s) = 1 \text{ and } q_2(s) = t_1(s) = t_2(s) = 0 \ \forall s \in S^2.$$

The proposed mechanism always assigns the good to bidder 1 for free. The construction of the mechanism immediately implies that it satisfies (1) EPIC, (2) IR, and (3) no subsidies. The mechanism is efficient because bidder 1's willingness to sell her unit (after winning the unit for free) exceeds her rival's willingness to pay for a unit for any signal realization,

$$d_1^0(0, s) \geq d_2^1(0, s) \ \forall s \in [1, 2]^2.$$

We prove that this inequality holds in the appendix. Bidder 1 does not have the highest willingness to pay for the good if $s_1 < s_2$; however, the auction assigns the unit to bidder 1, and it is efficient due to the strictly positive wealth effects. Bidder 1's willingness to sell the good after winning the good for a price below her willingness to pay (in this case for free) $d_1^0(0,s)$ exceeds her willingness to pay for the good $d_1^1(0,s)$, because she has strictly positive wealth effects. In the above example, the increase in bidder 1's willingness to sell increases her willingness to sell by a large enough amount to inhibit any Pareto improving resale opportunities.

In the next example, we study an English auction where bidders have classical preferences that do not satisfy weakly positive wealth effects. In particular, two bidders compete to win an inferior good (i.e. bidders have negative wealth effects). We show that the English auction has an ex post equilibrium where the bidder with the highest willingness to pay for the good always wins the good. However, the auction is inefficient.

Example 3. Suppose there are two bidders and $S_1 = S_2 = [0, \frac{1}{2}]$.

$$u_i^x(t,s) = (s_1 + (1-\epsilon)s_2)x + e^t,$$

where $\epsilon > 0$ is small. Bidder i's willingness to pay for the good is[17]

$$d_i^1(0,(s_i,s_j)) = -ln(1-(s_i+(1-\epsilon)s_j)).$$

The English auction has an ex post equilibrium where the bidder with the highest type always wins the good. Bidder i remains in the auction until the price reaches $b(s_i)$ where

$$b(s_i) = d_i^1(0,(s_i,s_i)) = -ln(1-(2-\epsilon)s_i).$$

The auction outcome is inefficient. To see this suppose that bidder 1 wins, and hence $s_1 > s_2$. Then bidder 1 wins and pays $b(s_2)$. A straightforward calculation shows that bidder 1 is willing to sell the good for a price of at least

$$ln(1+s_1-s_2) - ln(1-(2-\epsilon)s_2). \tag{3}$$

Bidder 2 is willing to buy the good for

$$-ln(1-(s_2+(1-\epsilon)s_1)). \tag{4}$$

We show in the appendix that for any $s_1 > s_2$, Expression (4) exceeds Expression (3) when ϵ is sufficiently small. Thus, the auction is inefficient because there is an ex post Pareto improving trade where the winning bidder sells the good to the losing bidder for a price

[17]See appendix for calculations related to this example.

above the winning bidder's willingness to sell and below the losing bidder's willingness to pay.

In the above example, the winning bidder pays a price that is below her willingness to pay. This makes the winning bidder feel wealthier, and because bidders have negative wealth effects, this lowers the price at which the winning bidder is willing to sell the good relative to her willingness to pay for the good. At the same time, the losing bidder's willingness to pay approximately equals the winning bidder's willingness to pay, because this is an almost common value setting. Hence, there are ex post Pareto improving trades between the two bidders.

In the next section, we show that the presence of strictly positive wealth effects similarly leads to inefficiencies in procurement auctions in the case where there is strong interdependence in bidder reservation costs, even when the winning supplier has the lowest reservation cost.

6 Efficient Procurement Auctions

In this section, we study the procurement setting $\mathbb{P}$. We obtain results that are qualitatively different from the results in Section 5 that related to mechanisms where an auctioneer sells a normal good. Recall that Theorem 1 showed that an auction is EPIC, IR, and efficient if the auction is IR and has an ex post equilibrium where the good is assigned to the bidder with the highest willingness to pay. Thus, assigning the good to the bidder with the highest willingness to pay is a sufficient condition for ex post Pareto efficiency when bidders have weakly positive wealth effects. However, in the procurement setting, we show that purchasing the good from the lowest cost bidder is necessary, but not sufficient, for ex post Pareto efficiency. In fact, we show that there is no auction with an efficient ex post equilibrium if bidders have strictly positive wealth effects and there is sufficiently strong interdependence in bidder preferences. The result holds even in the case where there is an auction with an ex post equilibrium where the lowest cost bidder is always the winning supplier.

6.1 Necessary and Sufficient Conditions

Lemma 4 below shows that if a mechanism $\Gamma^{\mathbb{P}}$ satisfies (1) EPIC, (2) IR, (3) no subsidies, and (4) efficiency, then the bidder with the lowest cost wins the auction. Note that Lemma 4 gives a necessary, but not sufficient condition for efficient implementation. Also, unlike the impossibility theorem presented later, we only need to assume weakly positive wealth effects to show this Lemma.

Lemma 4. *Consider a procurement setting and suppose that bidders have classical preferences that satisfy weakly positive wealth effects. If* $\Gamma^{\mathbb{P}}$ *satisfies (1) EPIC, (2) IR, (3) no subsidies, and (4) efficiency, then*

$$q_i(s) = 0 \implies d_i^0(0, s) \leq \min_{j \neq i} d_j^0(0, s), \forall s \in S^n, i \in \{1, \ldots, n\}.$$

The proof is by contradiction. Suppose that a mechanism $\Gamma^{\mathbb{P}}$ satisfies Properties (1)–(4) and that for some $s \in S^n$, bidder i is selected as the winning supplier even though bidder i does not have the lowest reservation cost. If bidder i is the winning supplier and is paid an amount exactly equal to her reservation cost $d_i^0(0, s)$, then bidder i has no unit and she would be willing to pay $d_i^1(d_i^0(0, s), s) = d_i^0(0, s)$ to buy a unit from her rival. This equality follows the definitions of d_i^0 and d_i^1 — bidder i is indifferent between having one unit and receiving zero transfers and having no unit and receiving transfer $d_i^0(0, s)$. Note also that the mechanism satisfies IR. Therefore, the winning bidder i is paid (weakly) more than her reservation cost of supplying a unit. Or equivalently, bidder i is paid her reservation cost and then given an additional non-negative payment. The additional payment makes bidder i wealthier. The increase in bidder i's wealth weakly increases the amount she is willing to pay for a unit due to the weakly positive wealth effects. Thus, the winning supplier (who now no longer owns a unit) is willing to pay a rival bidder at least $d_i^0(0, s)$ for a unit. If the winning bidder does not have the lowest reservation cost, then there is another bidder j who is willing to supply a unit for a price below bidder i's reservation cost. Hence, there is a Pareto improving trade where the winning bidder i pays the lowest cost bidder a price $p \in (\min_{j \neq i} d_j^0(0, s), d_i^0(0, s))$ to repurchase the good.

Lemma 4 illustrates differences between the sell setting and the procurement setting. In the sell setting, Example 2 shows that there is an auction that satisfies Properties (1)-(4) stated above, but the auction does not assign the good to the bidder with the highest willingness to pay. In addition, Theorem 1 shows that in the sell setting, an IR and ex post implementable mechanism that assigns the good to the highest willingness to pay bidder is efficient. In contrast to both points, Lemma 4 shows that in the procurement setting, purchasing the good from the lowest cost bidder is necessary, but not sufficient, condition for efficiency.

Thus, we know that if there is a mechanism that satisfies (1) EPIC, (2) IR, (3) no subsidies, and (4) efficiency, then the mechanism picks the lowest cost bidder to be the winning supplier. We use this observation to simplify our auction design problem. In particular, we define a class of candidate mechanisms for our procurement setting. A candidate mechanism satisfies two properties. First, it selects a lowest cost bidder as the winning supplier. Second, the payment rule is such that losing bidders are never paid and a winning bidder is paid the smallest amount that ensures EPIC and IR are always satisfied. In particular, the winning supplier is paid $\sup_{\tilde{s}_i \in S_i} d_i^0(0, (\tilde{s}_i, s_{-i}))$ s.t. $q_i(\tilde{s}_i, s_{-i}) = 0$. In words, the winner is paid the highest value of her reservation cost that is still associated with her being selected as the winning supplier.

Definition 8. A candidate mechanism $\Gamma^{\mathbb{P}*} = \{q, \tau\}$ has the following properties. It always selects the lowest cost bidder to be the winning supplier, i.e.,

$$q_i(s) = 0 \implies d_i^0(0, s) \leq \min_{j \neq i} d_j^0(0, s), \ \forall s = (s_1, \ldots, s_n) \in S^n, i \in \{1, \ldots, n\}.$$

For all $s \in S^n$ and all $i \in \{1, \ldots, n\}$, $\tau_i^1(s_{-i}) = 0$ and $\tau_i^0(s_{-i}) = \tau_i^{0*}(s_{-i})$ where

$$\tau_i^{0*}(s_{-i}) = \sup_{\tilde{s}_i \in S_i} d_i^0(0, (\tilde{s}_i, s_{-i})) \text{ s.t. } q_i(\tilde{s}_i, s_{-i}) = 0.$$

Note that $\tau_i^{0*}(s_{-i})$ is the lowest upper bound on the reservation cost that bidder i could have reported, under the constraints that (1) bidder i's rivals have types s_{-i} and (2) bidder i has the lowest reservation cost.

If the winning supplier is paid a lower amount, then there are cases where she is selected as the winning supplier but is paid below her reservation cost, which would violate IR. Alternatively, if we modify a candidate mechanism and instead pay the winner a greater amount, then we argue that the modified mechanism is efficient only if the relatively cheaper-to-run candidate mechanism is also efficient. Thus, we show that there exists a mechanism satisfying (1)–(4) if and only if there is a candidate $\Gamma^{\mathbb{P}*}$ mechanism that does (Lemma 5). We then proceed to use this necessary and sufficient condition to establish Theorem 2, which shows that there is no mechanism that satisfies (1)–(4) when the level of interdependence among bidders is sufficiently strong.

Lemma 5. *Consider a procurement setting and suppose that bidders have classical preferences that satisfy weakly positive wealth effects. There exists a mechanism that satisfies (1) IR, (2) EPIC, (3) no subsidies, and (4) efficiency if and only if a $\Gamma^{\mathbb{P}*}$ mechanism satisfies EPIC and efficiency.*

The proof can be understood intuitively. If there is a mechanism $\tilde{\Gamma}^{\mathbb{P}}$ that satisfies the four properties, then we show that there is a $\Gamma^{\mathbb{P}*}$ mechanism satisfying the four properties as well. The key step of the proof is to show that the outcome of a $\Gamma^{\mathbb{P}*}$ mechanism is efficient if the outcome of mechanism $\tilde{\Gamma}^{\mathbb{P}}$ is efficient. We show this in two steps. First, Lemma 4 implies that $\tilde{\Gamma}^{\mathbb{P}}$ selects a lowest cost bidder as the winning supplier, just as a $\Gamma^{\mathbb{P}*}$ mechanism does. Thus, there is a $\Gamma^{\mathbb{P}*}$ that has the same allocation rule as $\tilde{\Gamma}^{\mathbb{P}}$. Second, we show that any mechanism that satisfies the four properties must pay the winning bidder i at least $\tau_i^{0*}(s_{-i})$. To see this note that if bidder i supplies the good and is paid an amount p that is less than $\tau_i^{0*}(s_{-i})$, then there exists a $\tilde{s}_i$ such that bidder i has the lowest costs, yet bidder i is paid an amount less then her reservation cost $p < \tau_i^{0*}(s_{-i})$ and this would violate IR. Therefore, if $\tilde{\Gamma}^{\mathbb{P}}$ satisfies the four properties, it must pay the winning bidder a weakly greater amount than the amount the winning bidder is paid in a $\Gamma^{\mathbb{P}*}$ mechanism. In other words, the winning bidder is weakly richer under the outcome of $\tilde{\Gamma}^{\mathbb{P}}$. Thus, weakly positive wealth effects imply that the winning bidder (who ends the mechanism without a unit) is willing to pay one of her rival bidders a weakly greater amount for one of their units in the outcome induced by mechanism $\tilde{\Gamma}^{\mathbb{P}}$ as opposed to the outcome induced by a $\Gamma^{\mathbb{P}*}$ mechanism. Thus, if there are no ex post Pareto improving trades in the outcome implemented by $\tilde{\Gamma}^{\mathbb{P}}$, then there are no Pareto improving trades under $\Gamma^{\mathbb{P}*}$, because the winning bidder is weakly poorer, and hence less inclined to trade under the $\Gamma^{\mathbb{P}*}$ outcome.

In the next subsection, we study the efficiency of $\Gamma^{\mathbb{P}*}$ mechanisms in a symmetric setting where the outcome of a $\Gamma^{\mathbb{P}*}$ mechanism is equivalent to the outcome of an English auction.

6.2 A Symmetric Procurement Environment

In this section, we show that there is no mechanism that satisfies (1) EPIC, (2) IR, (3) no subsidies, and (4) efficiency when there is sufficiently strong interdependence in bidder preferences. We study a symmetric environment that is parameterized by $\alpha \in [0, 1]$, which measures the level of interdependence of bidder preferences. We show that the necessary and sufficient condition for efficient design from Lemma 5 is satisfied if and only if the level of interdependence α is sufficiently small. In particular, we show that all $\Gamma^{\mathbb{P}*}$ mechanisms are inefficient when the level of interdependence is sufficiently large. Note that our impossibility theorem will stipulate that bidders have classical preferences that satisfy strictly positive wealth effects. We impose the stricter assumption on bidder preferences in order to rule out the benchmark quasilinear setting, where there are well-studied necessary and sufficient conditions under which the English auction has an efficient ex post equilibrium.

We assume bidders have single-dimensional types, $S_i = [0, 1] \subset \mathbb{R},\ \forall i \in \{1, \ldots, n\}$, and there is a function U^1 such that

$$u_i^1(t, s) = U^1(t, s_i + \alpha \sum_{j \neq i} s_j),\ \forall t \in \mathbb{R},\ s \in [0, 1]^n,\ i \in \{1, \ldots, n\}, \alpha \in [0, 1],$$

where U^1 is continuous, strictly increasing in the first argument, and strictly decreasing in the second argument. There is also a function U^0 such that

$$u_i^0(t,s) = U^0(t),\ \forall t \in \mathbb{R},\ s \in [0,1]^n,\ i \in \{1,\dots,n\}, \alpha \in [0,1],$$

where U^0 is strictly increasing. As noted above, we continue to assume that $u = (u_1, \dots, u_n)$ represent classical preferences and also satisfy strictly positive wealth effects. If $\alpha = 0$, this is a pure private-value setting and if $\alpha = 1$ this is a pure common-value setting. For notational simplicity, we write bidder i's reservation cost of supplying a unit $d_i^0(0,s)$ as $c(s_i + \alpha \sum_{j \neq i} s_j) := d_i^0(0,s)$. Note that $c(\cdot)$ is strictly decreasing, because a bidder with a higher signal has a stronger incentive to sell her unit in exchange for money. Thus, the lowest cost bidder is the bidder with the highest signal. We say that bidders with these preferences have symmetric-additive preferences.

Definition 9. (Symmetric-additive preferences)

Bidders have symmetric-additive preferences if they have classical preferences with the property that there are functions U^0 and U^1 where

$$U^0(t) = u_i^0(t,s),\ \forall t \in \mathbb{R},\ s \in [0,1]^n,\ i \in \{1,\dots,n\},$$

$$U^1(t, s_i + \alpha \sum_{j \neq i} s_j) = u_i^1(t,s),\ \forall t \in \mathbb{R},\ s \in [0,1]^n,\ i \in \{1,\dots,n\}, \alpha \in [0,1].$$

Both U^0 and U^1 are continuously differentiable and strictly increasing in the first argument, while U^1 is continuous and strictly decreasing in the second argument.

Next, we derive conditions under which a $\Gamma^{\mathbb{P}*}$ mechanism satisfies Properties (1)–(4). Note that in this setting, a $\Gamma^{\mathbb{P}*}$ mechanism satisfies (1) EPIC, (2) IR, and (3) no subsidies by construction. Thus, a $\Gamma^{\mathbb{P}*}$ mechanism has an ex post equilibrium in which the lowest cost bidder always supplies her unit. However, a $\Gamma^{\mathbb{P}*}$ mechanism violates efficiency when the level of interdependence among bidders' preferences, α, is sufficiently large. Hence, Lemma 5 implies that there is a mechanism that satisfies Properties (1)–(4) if and only if α is sufficiently small. This is stated in Theorem 2 below.

Theorem 2. *Consider a procurement setting where bidders have symmetric-additive preferences that satisfy strictly positive wealth effects. There exists an $\alpha^* \in [0,1)$ such that a procurement mechanism satisfying (1) EPIC, (2) IR, (3) no subsidies, and (4) efficiency exists if and only if $\alpha \leq \alpha^*$.*

The intuition for the result can be understood by studying the extreme cases of pure common values ($\alpha = 1$) and pure private values ($\alpha = 0$) with two bidders.

In the pure common-value setting, fix $s \in S^n$ and suppose that bidder 1 is selected as the winning supplier by a $\Gamma^{\mathbb{P}*}$ mechanism. Bidder 1 is paid an amount that exceeds her reservation cost because a $\Gamma^{\mathbb{P}*}$ mechanism is individually rational.[18] In addition, since we are in a pure common value setting, bidder 2 has the same reservation cost as bidder 1. If bidder 1 were paid exactly her reservation cost, then she would be willing to pay bidder 2 an amount up to her reservation cost to purchase bidder 2's unit from her. This is because bidder 1 is indifferent between supplying her unit and not supplying her unit when she is paid exactly her reservation cost. Yet, bidder 1 is typically paid more than her reservation cost because she is compensated for her information rents in a mechanism that is EPIC. Thus, it is as though bidder 1 was paid her reservation cost and given some additional money. When bidder 1 is given additional money, she is willing to offer bidder 2 more money to repurchase the good, due to strictly positive wealth effects. Thus, bidder 1 is willing to pay bidder 2 an amount that strictly exceeds her reservation cost. In addition, bidder 2 is willing to supply bidder 1 with a unit when paid at least her reservation cost, $c(s_1 + s_2)$. Therefore, bidder 1 is willing to pay bidder 2 an amount that strictly exceeds bidder 2's reservation cost in order to buy her unit. Thus, there is an ex post Pareto improving trade where bidder 1 (who sold her unit to the auctioneer) buys bidder 2's unit. Thus the $\Gamma^{\mathbb{P}*}$ mechanism is inefficient, and Lemma 5 implies that there is no mechanism that satisfies the four desired properties in this setting.

In contrast, when bidders have pure private values ($\alpha = 0$), any $\Gamma^{\mathbb{P}*}$ mechanism is efficient (and equivalent to a second price auction). We can see the intuition by again considering a two-bidder setting where $s_1 > s_2$, and hence bidder 1 is selected as the winning supplier by a $\Gamma^{\mathbb{P}*}$ mechanism. Bidder 1 is paid her rival's reservation cost $c(s_2)$, where $c(s_2) > c(s_1)$. Therefore, bidder 1 is paid an amount that exceeds her reservation cost, and her utility increases.[19] In this case, there are no ex post Pareto improving trades between the two bidders. To see this, note that bidder 2 is only willing to sell her unit if she is paid at least $c(s_2)$. However, if bidder 1 pays bidder 2 an amount $p \geq c(s_2)$, her wealth decreases (weakly) relative to her wealth prior to the auction, because she pays bidder 2 an amount p that exceeds the amount she was paid in the auction $c(s_2)$. Thus, any acceptable offer to bidder 2 makes bidder 1 worse off and there are no Pareto improving trades. Therefore, any $\Gamma^{\mathbb{P}*}$ mechanism is efficient in the private value setting.

Note that in our symmetric setting, a $\Gamma^{\mathbb{P}*}$ mechanism is implemented by an English auction with a particular tie-breaking rule. Thus, we can say that there is an efficient auction in our symmetric procurement if and only if the English auction is efficient.

[18] More formally, bidder 1 is paid $\tau_i^{0*}(s_2)$ where $\tau_i^{0*}(s_2) = c(2s_2) > c(s_1 + s_2)$ if $s_1 > s_2$.

[19] $c(s_2) > c(s_1) \implies u^0(c(s_2), s_1) > u^0(c(s_1), s_1) = u^1(0, s_1) = 0$ where the first equality follows from the definition of $c(\cdot)$.

Corollary 3. *Consider a procurement setting where bidders have symmetric-additive preferences. An English auction has a symmetric ex post equilibrium that is equivalent to the outcome of a $\Gamma^{\mathbb{P}*}$ mechanism.*

Corollary 3 follows from Lemma 2 and Theorem 10 in Milgrom and Weber (1982). Milgrom and Weber (1982) show that in a symmetric quasilinear setting, the English auction has an ex post equilibrium where the high value bidder (here lowest cost) wins the object. Lemmas 2 and 3 then imply that the English auction satisfies EPIC and IR in the non-quasilinear setting because it satisfies the both conditions in the corresponding quasilinear setting. Thus, we see that the English auction has an ex post equilibrium where the winning supplier is the bidder with the lowest cost. However, when the degree of interdependence among bidders is sufficiently strong, the outcome of the English auction is inefficient in the non-quasilinear setting where bidders have strictly positive wealth effects.

The intuition for the impossibility result given above is different from intuition used to prove the impossibility results in other interdependent value auction settings where bidders have quasilinear preferences. The impossibility theorems given by Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001), and Jehiel et al. (2006) are all related to interdependent value settings where bidders have quasilinear preferences and multi-dimensional private information. The efficient social choice function is a social choice function that assigns the good to the lowest cost (or highest willingness to pay) bidder. Those papers show that the efficient social choice function is not ex post implementable. Bergemann and Morris (2009) also present an impossibility result related to implementation in a single unit auction setting where bidders are quasilinear and have interdependent and additive values (Section 7). They show that robust implementation is possible if and only if the interdependence in bidder preferences is not too strong. Bergemann and Morris' notion of robust implementation is stronger than EPIC, and implementation fails because truthful reporting is not the unique solution to iterative elimination of never best replies when bidder interdependence is sufficient strong. In all of these papers, efficient implementation fails because there is no mechanism that implements an assignment where the bidder with the highest valuation (lowest cost) wins.

In contrast, we study a setting where there is an ex post equilibrium where the lowest cost bidder is selected as the winning supplier. That would be necessary and sufficient for efficient implementation with quasilinearity. However, without quasilinearity, the space of efficient outcomes is different and assigning the good to the lowest cost bidder is necessary, but not sufficient, for efficient auction design. Instead, we show that strictly positive wealth effects make the winning bidder more inclined to buy a unit from one of her rival suppliers, and this leads to the existence of Pareto improving trades following the auction when there is strong interdependence.

7 Conclusion

We study efficient and ex post implementable mechanisms when bidders have non-quasilinear preferences and positive wealth effects. Our setting nests well-studied cases such as auctions with risk averse bidders, auctions with financially constrained bidders, and auctions for risky assets. We study the incentive and efficiency properties of mechanisms in the non-quasilinear setting by considering the mechanism's corresponding quasilinear setting.

We show that in the canonical auction setting where the auctioneer sells a good, a mechanism is efficient in the non-quasilinear setting if the mechanism is efficient in the corresponding quasilinear setting. This implies that conditions guaranteeing the efficiency of the English auction in a quasilinear setting translate to the non-quasilinear setting when bidders have weakly positive wealth effects. Moreover, this gives us a simple method for computing equilibrium bid behavior in English auctions without quasilinearity. We get distinct results in procurement settings. We show that there are cases where there is no mechanism that has desirable incentive and efficiency properties, even if there is an ex post implementable mechanism where the winning supplier is the bidder with the lowest cost.

The methodology used in this paper could also be useful in further research on other mechanism design settings without quasilinearity. More precisely, we would be interested in studying how we could construct corresponding quasilinear settings in other mechanism design problems where agents have non-quasilinear preferences. We think that this could help us to understand the limits of efficient and ex post implementation without quasilinearity in multi-unit auctions, combinatorial assignment problems, or double auction models.

8 Proofs

Calculations for Footnote 3: We show that $c = e^{s_1-s_2} - e^{-(1+\alpha)s_2}$. Recall we define c as solving

$$s_1 + \alpha s_2 + ln(1 - d(s_2, s_2)) = ln(1 - d(s_2, s_2) + c). \tag{5}$$

Noting that $d(s_2, s_2) = 1 - e^{-(1+\alpha)s_2}$, the LHS of Equation (5) simplifies to $s_1 + \alpha s_2 - (1+\alpha)s_2 = s_1 - s_2$, and the RHS is $ln(e^{-(1+\alpha)s_2} + c)$. Thus, Equation (5) states

$$s_1 - s_2 = ln(e^{-(1+\alpha)s_2} + c) \implies c = e^{s_1-s_2} - e^{-(1+\alpha)s_2}.$$

Calculations for Footnote 4: We show that bidder 1's willingness to sell after winning, $c = e^{s_1-s_2} - e^{-(1+\alpha)s_2}$, weakly exceeds her willingness to pay $d(s_1, s_2) = 1 - e^{-(s_1+\alpha s_2)}$ when $s_1 \geq s_2$. Let $\mathcal{C}(s_1, s_2) = e^{s_1-s_2} - e^{-(1+\alpha)s_2}$. Note that $\mathcal{C}(s_2, s_2) = d(s_2, s_2)$. In addition, when $s_1 > s_2 \geq 0$, then

$$s_1 - s_2 > -(s_1 + \alpha s_2) \implies \frac{\partial \mathcal{C}(s_1, s_2)}{\partial s_1} = e^{s_1-s_2} > e^{-(s_1+\alpha s_2)} = \frac{\partial d(s_1, s_2)}{\partial s_1}.$$

Thus $s_1 \geq s_2 \implies \mathcal{C}(s_1, s_2) = e^{s_1-s_2} - e^{-(1+\alpha)s_2} \geq 1 - e^{-(s_1+\alpha s_2)} = d(s_1, s_2)$, because the two quantities are equal when $s_1 = s_2$. The LHS strictly exceeds the RHS when $s_1 > s_2$.

Calculations for Footnote 6: Recall $c(s_i, s_j) = e^{(s_i+\alpha s_j)} - 1$. Bidder 1 is paid $c(s_2, s_2) = e^{s_2(1+\alpha)} - 1$ to supply her unit. Thus, she has utility $ln(e^{s_2(1+\alpha)}) = s_2(1 + \alpha)$. She is willing to pay at most d to repurchase a unit, where d is solves

$$s_2(1+\alpha) = s_1 + \alpha s_2 + ln(e^{s_2(1+\alpha)} - d) \implies s_2 - s_1 = ln(e^{s_2(1+\alpha)} - d) \implies e^{s_2(1+\alpha)} - e^{s_2-s_1} = d.$$

Calculations for Footnote 7: Fix $s_1 < s_2$. We show that

$$d = e^{(1+\alpha)s_2} - e^{s_2-s_1} > e^{s_2+\alpha s_1} - 1 = c(s_2, s_1).$$

First, we show that the above inequality holds when $\alpha = 1$. Hence, we show that if $s_1 < s_2$, then

$$e^{2s_2} - e^{s_2-s_1} > e^{s_2+s_1} - 1 = c(s_2, s_1).$$

Let $\mathcal{D}(s_1, s_2) = e^{2s_2} - e^{s_2-s_1}$. Note that

$$\frac{\partial \mathcal{D}(s_1, s_2)}{\partial s_1} = e^{s_2-s_1} < e^{s_2+s_1} = \frac{\partial c(s_2, s_1)}{\partial s_1},$$

and

$$\mathcal{D}(s_2, s_2) = e^{2s_2} - 1 = c(s_2, s_2).$$

Therefore, $\mathcal{D}(s_1, s_2) > c(s_2, s_1)$ if $s_1 < s_2$. This also implies that

$$e^{(1+\alpha)s_2} - e^{s_2 - s_1} > e^{s_2 + \alpha s_1} - 1$$

when α is sufficiently close to 1 because both sides of the inequality are continuous in α and the inequality holds when $\alpha = 1$. Thus, when α is sufficiently large there is an ex post Pareto improving trade where bidder 1 buys a unit from bidder 2 for price p where,

$$d = e^{(1+\alpha)s_2} - e^{s_2 - s_1} > p > e^{s_2 + \alpha s_1} - 1 = c(s_2, s_1).$$

Calculations for Example 2: First, we show $d_i^1(0, s) = 1 - e^{-(s_i + \frac{1}{2}s_j)}$. Note we define d_i^1 as solving

$$u_i^0(0, s) = 0 = u_i^1(-d_i^1(0, s), s) = s_i + \frac{1}{2}s_j + ln(1 - d_i^1(0, s))$$

$$\implies e^{-(s_i + \frac{1}{2}s_j)} = 1 - d_i^1(0, s) \implies d_i^1(0, s) = 1 - e^{-(s_i + \frac{1}{2}s_j)}.$$

We show that $d_i^0(t, s) = (1 + t)(e^{s_i + \frac{1}{2}s_j} - 1)$ because d_i^0 is such that

$$u_i^1(t, s) = s_i + \frac{1}{2}s_j + ln(1+t) = u_i^0(t + d_i^0(t, s), s) = ln(1 + t + d_i^0(t, s)) \implies e^{(s_i + \frac{1}{2}s_j + ln(1+t))} = 1 + t + d_i^0(t, s)$$

$$\implies (1 + t)e^{s_i + \frac{1}{2}s_j} = 1 + t + d_i^0(t, s) \implies d_i^0(t, s) = (1 + t)(e^{s_i + \frac{1}{2}s_j} - 1).$$

The proposed mechanism is efficient if bidder 1's willingness to sell conditional on winning a unit for free exceeds bidder 2's willingness to pay for all $s \in [1, 2]^2$:

$$d_1^0(0, s) = e^{s_1 + \frac{1}{2}s_2} - 1 \geq 1 - e^{-(s_2 + \frac{1}{2}s_1)} = d_2^1(0, s) \ \forall s \in [1, 2]^2.$$

Equivalently the outcome is efficient if

$$e^{s_1 + \frac{1}{2}s_2} + e^{-(s_2 + \frac{1}{2}s_1)} - 2 \geq 0 \ \forall (s_1, s_2) \in [1, 2]^2.$$

The above inequality holds (with a strict inequality) because

$$e^{s_1 + \frac{1}{2}s_2} + e^{-(s_2 + \frac{1}{2}s_1)} - 2 > e - 2 > 0 \ \forall (s_1, s_2) \in [1, 2]^2,$$

where the first inequality follows because $e^{s_1 + \frac{1}{2}s_2} \geq e$ and $e^{-(s_2 + \frac{1}{2}s_1)} > 0 \ \forall (s_1, s_2) \in [1, 2]^2$.

Thus, we have shown that the proposed mechanism is efficient, because the winning bidder's willingness to sell following the mechanism exceeds her rival's willingness to pay for any possible signal realization.

Calculations for Footnote 16: Bidder i's willingness to pay is $d_i^1(0,(s_i,s_j))$ where

$$s_i + (1-\epsilon)s_j + e^{-d_i^1(0,(s_i,s_j))} = e^0 = 1 \implies d_i^1(0,(s_i,s_j)) = -ln(1-(s_i+(1-\epsilon)s_j)).$$

Fix $s \in S^2$. Suppose that $s_1 > s_2 > 0$. We show that the ex post equilibrium outcome of the English auction is inefficient when $\epsilon > 0$ is sufficiently small. Since $s_1 > s_2$, bidder 1 wins the unit in the auction and pays $b(s_2) = -ln(1-(2-\epsilon)s_2)$. Her willingness to sell after winning equals h where

$$s_1 + (1-\epsilon)s_2 + e^{-b(s_2)} = e^{h-b(s_2)} \implies s_1 + (1-\epsilon)s_2 + 1 - (2-\epsilon)s_2 = e^{h+ln(1-(2-\epsilon)s_2)}$$

$$\implies ln(1+s_1-s_2) = h + ln(1-(2-\epsilon)s_2) \implies h = ln(1+s_1-s_2) - ln(1-(2-\epsilon)s_2).$$

The English auction outcome is inefficient if bidder 1's willingness to sell after winning, h, is strictly below her rival's willingness to pay. This occurs if

$$d_2^1(0,(s_2,s_1)) = -ln(1-(s_2+(1-\epsilon)s_1)) > ln(1+s_1-s_2) - ln(1-(2-\epsilon)s_2) = h.$$

Let $R(s_1,s_2) : ln(1+s_1-s_2) - ln(1-(2-\epsilon)s_2)$ be the right hand side above. Note that $d_2^1(0,(s_2,s_1)) = R(s_1,s_2)$ if $s_1 = s_2$. In addition,

$$\frac{\partial d_2^1(0,(s_2,s_1))}{\partial s_1} = \frac{1-\epsilon}{1-(s_2+(1-\epsilon)s_1)} > \frac{1}{1+s_1-s_2} = \frac{\partial R(s_1,s_2)}{\partial s_1}$$

when ϵ is sufficiently small because the denominator of the LHS is strictly less than the denominator of the RHS. Therefore,

$$s_1 > s_2 \implies d_2^1(0,(s_2,s_1)) > R(s_1,s_2).$$

Thus, the outcome of the English auction is inefficient when $\epsilon > 0$ is sufficiently small, because there is a Pareto improving trade where bidder 1 sells her unit to bidder 2 for price p that is between bidder 2's willingness to pay $d_2^1(0,(s_2,s_1))$ and bidder 1's willingness to sell $R(s_1,s_2)$.

Proof of Lemma 1

The proof is by contradiction. Suppose Γ^c satisfies EPIC. Fix $s_{-i} \in S^{-i}$ and suppose that there exist $s_i', s_i'' \in S_i$ such that $q_i(s_i', s_{-i}) = q_i(s_i'', s_{-i})$, yet $t_i(s_i', s_{-i}) < t_i(s_i'', s_{-i})$ for some

$i \in \{1, \ldots, n\}$. Then

$$u_i^{q_i(s_i', s_{-i})}(t_i(s_i', s_{-i}), (s_i', s_{-i})) < u_i^{q_i(s_i'', s_{-i})}(t_i(s_i'', s_{-i}), (s_i', s_{-i})).$$

This violates EPIC. In words, it is not a best response for bidder i to report s_i' given her rivals report s_{-i}. This is because bidder i wins the same number of units and receives a greater transfer (or equivalently makes a lower payment) when she reports that her signal is s_i'' instead. That violates EPIC. Thus, EPIC implies that $t_i(s_i, s_{-i}) = t_i(s_i', s_{-i})$ if $q_i(s_i, s_{-i}) = q_i(s_i', s_{-i})$.

Proof of Lemma 2:

Recall a mechanism $\Gamma^c = \{q, \tau\}$ is EPIC in the non-quasilinear setting if $\forall i \in \{1, \ldots, n\}$, $s, s' \in S^n$ s.t. $s = (s_i, s_{-i})$ and $s = (s_i', s_{-i})$:

$$u_i^{q_i(s)}(\tau_i^{q_i(s)}(s_{-i}), s) \geq u_i^{q_i(s')}(\tau_i^{q_i(s')}(s_{-i}), s), \tag{6}$$

and EPIC in the quasilinear setting $QL(\Gamma^c, u)$ if $\forall i \in \{1, \ldots, n\}$, $s, s' \in S^n$ s.t. $s = (s_i, s_{-i})$ and $s = (s_i', s_{-i})$:

$$q_i(s)v_i(s) + \tau_i^{q_i(s)}(s_{-i}) \geq q_i(s')v_i(s) + \tau_i^{q_i(s')}(s_{-i}), \tag{7}$$

where $v_i(s) = d_i^{1-\omega^c}(\tau_i^{\omega^c}(s_{-i}), s)$.

In our proof, we show that (6) is satisfied in the non-quasilinear if and only if (7) is satisfied in setting $QL(\Gamma^c, u)$. In particular, we show that (6) holds if and only if (7) holds by considering three cases.

Case 1: Suppose that $s_i, s_i' \in S_i$ and $s_{-i} \in S^{-i}$ are such that $q_i(s_i, s_{-i}) = q_i(s_i', s_{-i})$. Then (6) and (7) both bind with equality because Remark **??** shows that $t_i(s_i, s_{-i}) = t_i(s_i', s_{-i}) = \tau_i^{q_i(s_i, s_{-i})}(s_{-i})$. Thus,

$$q_i(s) = q_i(s') \implies u_i^{q_i(s)}(\tau_i^{q_i(s)}(s_{-i}), s) = u_i^{q_i(s')}(\tau_i^{q_i(s')}(s_{-i}), s),$$

and

$$q_i(s) = q_i(s') \implies q_i(s)v_i(s) + \tau_i^{q_i(s)}(s_{-i}) = q_i(s')v_i(s) + \tau_i^{q_i(s')}(s_{-i}),$$

where $s = (s_i, s_{-i})$ and $s' = (s_i', s_{-i})$. Thus, (6) and (7) both hold trivially when $s_i, s_i' \in S_i$ and $s_{-i} \in S^{-i}$ are such that $q_i(s_i, s_{-i}) = q_i(s_i', s_{-i})$.

Case 2: Suppose that $s_i, s_i' \in S_i$ and $s_{-i} \in S^{-i}$ are such that $q_i(s_i, s_{-i}) = 1 - \omega^c$ and $q_i(s_i', s_{-i}) = \omega^c$. Then, mechanism Γ^c satisfies (6) in the non-quasilinear setting if and only if

$$u_i^{1-\omega^c}(\tau_i^{1-\omega^c}(s_{-i}), s) \geq u_i^{\omega^c}(\tau_i^{\omega^c}(s_{-i}), s). \tag{8}$$

Note that we can rewrite the RHS of Inequality (8) as

$$u_i^{\omega^c}(\tau_i^{\omega^c}(s_{-i}), s) = u_i^{1-\omega^c}(\tau_i^{\omega^c}(s_{-i}) + (2\omega^c - 1)d_i^{1-\omega^c}(\tau_i^{\omega^c}(s_{-i}), s), s). \tag{9}$$

To see why, consider the sell case and procurement cases separately. If $c = \mathbb{S}$, then the definition of d_i^1 implies that

$$u_i^0(\tau_i^0(s_{-i}), s) = u_i^1(\tau_i^0(s_{-i}) - d_i^1(\tau_i^0(s_{-i}), s), s).$$

Since $\omega^{\mathbb{S}} = 0$, (9) holds. Similarly, if $c = \mathbb{P}$, the definition of d_i^0 implies

$$u_i^1(\tau_i^1(s_{-i}), s) = u_i^0(\tau_i^1(s_{-i}) + d_i^0(\tau_i^1(s_{-i}), s), s).$$

Since $\omega^{\mathbb{P}} = 1$, we again get that (9) holds.

Thus, by substituting Equation (9) into Expression (8), we can say that mechanism Γ^c satisfies Inequality (6) in the non-quasilinear setting if and only if

$$u_i^{1-\omega^c}(\tau_i^{1-\omega^c}(s_{-i}), s) \geq u_i^{1-\omega^c}(\tau_i^{\omega^c}(s_{-i}) + (2\omega^c - 1)d_i^{1-\omega^c}(\tau_i^{\omega^c}(s_{-i}), s), s).$$

This inequality holds if and only if

$$\tau_i^{1-\omega^c}(s_{-i}) \geq \tau_i^{\omega^c}(s_{-i}) + (2\omega^c - 1)d_i^{1-\omega^c}(\tau_i^{\omega^c}(s_{-i}), s),$$

because $u_i^{1-\omega^c}$ is strictly increasing in its first argument. Rearranging this expression yields

$$(1 - \omega^c)d_i^{1-\omega^c}(\tau_i^{\omega^c}(s_{-i}), s) + \tau_i^{1-\omega^c}(s_{-i}) \geq \omega^c d_i^{1-\omega^c}(\tau_i^{\omega^c}(s_{-i}), s) + \tau_i^{\omega^c}(s_{-i}).$$

Recall that in this case, $q_i(s_i, s_{-i}) = 1 - \omega^c$, $q_i(s_i', s_{-i}) = \omega^c$, and $v_i(s) = d_i^{1-\omega^c}(\tau_i^{\omega^c}(s_{-i}), s)$. Thus, the condition in (8) holds if and only if

$$q_i(s_i, s_{-i})v_i(s) + \tau_i^{q_i(s_i,s_{-i})}(s_{-i}) \geq q_i(s_i', s_{-i})v_i(s) + \tau_i^{q_i(s_i',s_{-i})}(s_{-i}). \tag{10}$$

Thus, we have shown that for any $s_i, s_i' \in S_i$ and $s_{-i} \in S^{-i}$ such that $q_i(s_i, s_{-i}) = 1 - \omega^c$ and $q_i(s_i', s_{-i}) = \omega^c$, mechanism Γ^c satisfies Inequality (6) in the non-quasilinear setting if and only if mechanism Γ^c satisfies Inequality (7) in the setting $QL(\Gamma^c, u)$.

Case 3: Suppose that $s_i, s_i' \in S_i$ and $s_{-i} \in S^{-i}$ are such that $q_i(s_i, s_{-i}) = \omega^c$ and $q_i(s_i', s_{-i}) = 1 - \omega^c$. Then mechanism Γ^c satisfies Inequality (6) in the non-quasilinear setting if and only if

$$u_i^{\omega^c}(\tau_i^{\omega^c}(s_{-i}), s) \geq u_i^{1-\omega^c}(\tau_i^{1-\omega^c}(s_{-i}), s). \tag{11}$$

Recall that we proved in Case 2 that the LHS of Expression (11) can be written as

$$u_i^{\omega^c}(\tau_i^{\omega^c}(s_{-i}), s) = u_i^{1-\omega^c}(\tau_i^{\omega^c}(s_{-i}) + (2\omega^c - 1)d_i^{1-\omega^c}(\tau_i^{\omega^c}(s_{-i}), s), s),$$

as in Equation (9). Thus, we rewrite Expression (11) to state that mechanism Γ^c satisfies Inequality (6) in the non-quasilinear setting if and only if

$$u_i^{1-\omega^c}(\tau_i^{\omega^c}(s_{-i}) + (2\omega^c - 1)d_i^{1-\omega^c}(\tau_i^{\omega^c}(s_{-i}), s), s) \geq u_i^{1-\omega^c}(\tau_i^{1-\omega^c}(s_{-i}), s).$$

We know that the above inequality holds if and only if

$$\tau_i^{\omega^c}(s_{-i}) + (2\omega^c - 1)d_i^{1-\omega^c}(\tau_i^{\omega^c}(s_{-i}), s) \geq \tau_i^{1-\omega^c}(s_{-i}),$$

because $u_i^{1-\omega^c}$ is strictly increasing in its first argument. Rearranging this expression yields

$$\omega^c d_i^{1-\omega^c}(\tau_i^{\omega^c}(s_{-i}), s) + \tau_i^{\omega^c}(s_{-i}) \geq (1 - \omega^c)d_i^{1-\omega^c}(\tau_i^{\omega^c}(s_{-i}), s) + \tau_i^{1-\omega^c}(s_{-i}).$$

Recall that in this case $q_i(s_i, s_{-i}) = \omega^c$, $q_i(s_i', s_{-i}) = 1 - \omega^c$, and $v_i(s) = d_i^{1-\omega^c}(\tau_i^{\omega^c}(s_{-i}), s)$. Thus the above inequality holds if and only if

$$q_i(s_i, s_{-i})v_i(s) + \tau_i^{q_i(s_i,s_{-i})}(s_{-i}) \geq q_i(s_i', s_{-i})v_i(s) + \tau_i^{q_i(s_i',s_{-i})}(s_{-i}). \tag{12}$$

Note that Inequality (12) is identical to Inequality (7) in this case. Thus, we have shown that for any $s_i, s_i' \in S_i$ and $s_{-i} \in S^{-i}$ such that $q_i(s_i, s_{-i}) = \omega^c$ and $q_i(s_i', s_{-i}) = 1 - \omega^c$, mechanism Γ^c satisfies Inequality (6) in the non-quasilinear setting if and only if mechanism Γ^c satisfies Inequality (7) in the setting $QL(\Gamma^c, u)$.

The three cases combine to show that Inequality (6) holds if and only if Inequality (7) for all $s \in S^n$. Hence, Γ^c is EPIC in the non-quasilinear setting if and only if it is EPIC in setting $QL(\Gamma^c, u)$.

Proof of Lemma 3

First we show that if mechanism Γ^c is EPIC and IR in our non-quasilinear setting, then Γ^c satisfies IR in setting $QL(\Gamma^c, u)$ (the $\implies$ direction). We prove the reverse direction ($\impliedby$) second.

Recall that IR in the non-quasilinear setting implies that

$$u_i^{q_i(s)}(t_i(s), s) \geq u_i^{\omega^c}(0, s), \tag{13}$$

for all $i \in \{1, \ldots, n\}$ and $s \in S^n$. Similarly, a mechanism satisfies IR in the corresponding quasilinear setting $QL(\Gamma^c, u)$ if for all $i \in \{1, \ldots, n\}$ and $s \in S^n$,

$$q_i(s)v_i(s) + \tau_i^{q_i(s)}(s_{-i}) \geq \omega^c v_i(s). \tag{14}$$

($\Longrightarrow$) We assume that Γ^c is EPIC and IR in the non-quasilinear setting. In our proof, we show bidder i's IR constraint is satisfied for all $s \in S^n$ in the setting $QL(\Gamma^c, u)$. We consider three possible cases (1, 2 and 3) and show IR is satisfied for any bidder $i \in \{1, \ldots, n\}$ case by case.

Case 1: Suppose $s \in S^n$ is such that $q_i(s_i, s_{-i}) = \omega^c$. Expression (13) implies that

$$u_i^{\omega^c}(\tau_i^{\omega^c}(s_{-i}), s) \geq u_i^{\omega^c}(0, s) \iff \tau_i^{\omega^c}(s_{-i}) \geq 0. \tag{15}$$

Then, mechanism Γ^c satisfies IR in setting $QL(\Gamma^c, u)$ because

$$\tau_i^{\omega^c}(s_{-i}) \geq 0 \iff \omega^c v_i(s) + \tau_i^{\omega^c}(s_{-i}) \geq \omega^c v_i(s). \tag{16}$$

Therefore, mechanism Γ^c satisfies IR in setting $QL(\Gamma^c, u)$ when s is such that $q_i(s) = \omega^c$.

Case 2: Let $s = (s_i, s_{-i}) \in S^n$ and suppose $s_{-i} \in S^{-i}$ is such that $q_i(\tilde{s}_i, s_{-i}) = 1 - \omega^c$, $\forall \tilde{s}_i \in S_i$. Thus, $s_{-i} \in S^{-i}$ is such that bidder i receives $1 - \omega^c$ for any type that she reports. In this case, Expression (13) is equivalent to

$$u_i^{1-\omega^c}(\tau_i^{1-\omega^c}(s_{-i}), s) \geq u_i^{\omega^c}(0, s) \iff \tau_i^{1-\omega^c}(s_{-i}) \geq (2\omega^c - 1)d_i^{1-\omega^c}(0, s). \tag{17}$$

To see why, we first consider the sell setting. Recall that $\omega^{\mathbb{S}} = 0$ in the sell setting $\mathbb{S}$ and our definition of $d_i^1(0, s)$ implies that

$$u_i^1(\tau_i^1(s_{-i}), s) \geq u_i^0(0, s) \iff \tau_i^1(s_{-i}) \geq -d_i^1(0, s) = (2\omega^{\mathbb{S}} - 1)d_i^{1-\omega^{\mathbb{S}}}(0, s),$$

where the final equality follows because $\omega^{\mathbb{S}} = 0$. Similarly, in the procurement setting $\mathbb{P}$, our definition of $d_i^0(0, s)$ implies that

$$u_i^0(\tau_i^0(s_{-i}), s) \geq u_i^1(0, s) \iff \tau_i^0(s_{-i}) \geq d_i^0(0, s) = (2\omega^{\mathbb{P}} - 1)d_i^{1-\omega^{\mathbb{P}}}(0, s),$$

where the final equality follows because $\omega^{\mathbb{P}} = 1$.

Next, recall that in Expressions (1) and (2) of Corollary 1 we set $\tau^{\omega^c}(s_{-i}) = 0$ because $q_i(\tilde{s}_i, s_{-i}) \neq \omega^c \ \forall \tilde{s}_i \in S_i$. Then Definition 7 states that bidder i's value in the corresponding quasilinear setting is $v_i(s) = d_i^{1-\omega^c}(0, s)$. When we substitute this into Expression (17) we have that

$$u_i^{1-\omega^c}(\tau_i^{1-\omega^c}(s_{-i}), s) \geq u_i^{\omega^c}(0, s) \iff \tau_i^{1-\omega^c}(s_{-i}) \geq (2\omega^c - 1)v_i(s). \tag{18}$$

By noting that $q_i(s) = 1 - \omega^c$, we have that

$$\tau_i^{1-\omega^c}(s_{-i}) \geq (2\omega^c - 1)v_i(s) \iff q_i(s)v_i(s) + \tau_i^{q_i(s)}(s_{-i}) \geq \omega^c v_i(s). \tag{19}$$

Combining (17) and (19) shows that

$$u_i^{1-\omega^c}(\tau_i^{1-\omega^c}(s_{-i}), s) \geq u_i^{\omega^c}(0, s) \iff q_i(s)v_i(s) + \tau_i^{q_i(s)} \geq \omega^c v_i(s). \tag{20}$$

Thus, we show that mechanism Γ^c satisfies IR in setting $QL(\Gamma^c, u)$ when $s_{-i} \in S^{-i}$ is such that $q_i(\tilde{s}_i, s_{-i}) = 1 - \omega^c$, $\forall \tilde{s}_i \in S_i$.

Case 3: Suppose that $s = (s_i, s_{-i}) \in S^n$ is such that $q_i(s) = 1 - \omega^c$ and that there is a $s_i' \in S_i$ such that $q_i(s_i', s_{-i}) = \omega^c$. Thus, we consider the case where $s_{-i} \in S^{-i}$ is such that bidder i is able to influence the number of units that she receives with her report, in contrast to Case 2.

Recall we assume that mechanism Γ^c is EPIC in the non-quasilinear setting. Lemma 2 then implies that mechanism Γ^c is EPIC in setting $QL(\Gamma^c, u)$. Therefore,

$$(1 - \omega^c)v_i(s) + \tau_i^{1-\omega^c}(s_{-i}) \geq \omega^c v_i(s) + \tau_i^{\omega^c}(s_{-i}),$$

where the LHS is bidder i's payoff when she truthfully reports her type and the RHS is bidder i's payoff when she reports type s_i' and wins $q_i(s_i', s_{-i}) = \omega^c$ units. Recall that Case 1 shows that $\tau_i^{\omega^c}(s_{-i}) \geq 0$. This follows under the assumption that IR holds in the non-quasilinear setting when profile of bidder signals are (s_i', s_{-i}). Thus,

$$\tau_i^{\omega^c}(s_{-i}) \geq 0 \implies (1 - \omega^c)v_i(s) + \tau_i^{1-\omega^c}(s_{-i}) \geq \omega^c v_i(s) + \tau_i^{\omega^c}(s_{-i}) \geq \omega^c v_i(s).$$

Equivalently,

$$q_i(s)v_i(s) + \tau_i^{q_i(s)} \geq \omega^c v_i(s),$$

which shows that mechanism Γ^c is IR in setting $QL(\Gamma^c, u)$ when $s = (s_i, s_{-i}) \in S^n$ is such that $q_i(s) = 1 - \omega^c$, and $\exists s_i' \in S_i$ such that $q_i(s_i', s_{-i}) = \omega^c$.

Cases 1, 2, and 3 combine to show that mechanism Γ^c satisfies IR in setting $QL(\Gamma^c, u)$ for all $s \in S^n$, when we assume mechanism Γ^c satisfies EPIC and IR in the non-quasilinear setting.

($\Longleftarrow$) For the reverse direction, we show that if mechanism Γ^c is IR in the setting $QL(\Gamma^c, u)$, then Γ^c is IR in the non-quasilinear setting. In addition, the statement of the Lemma assumes that Γ^c is EPIC in the non-quasilinear setting. The proof is nearly identical to the forward direction. We consider three analogous cases (4, 5 and 6).

Case 4: Suppose $s \in S^n$ is such that $q_i(s_i, s_{-i}) = \omega^c$. Expressions (15) and (16) in Case 1 show that

$$\omega^c v_i(s) + \tau_i^{\omega^c}(s_{-i}) \geq \omega^c v_i(s) \iff \tau_i^{\omega^c}(s_{-i}) \geq 0 \iff u_i^{\omega^c}(\tau_i^{\omega^c}(s_{-i}), s) \geq u_i^{\omega^c}(0, s).$$

Thus, the mechanism is IR in our non-quasilinear setting (i.e. satisfies the in right most inequality), because we assume the mechanism is IR in setting $QL(\Gamma^c, u)$ (i.e. satisfies the left inequality).

Case 5: Suppose that $s = (s_i, s_{-i}) \in S^n$ and $s_{-i} \in S^{-i}$ is such that $q_i(\tilde{s}_i, s_{-i}) = 1 - \omega^c$, $\forall \tilde{s}_i \in S_i$. Thus, s_{-i} is such that bidder i receives $1 - \omega^c$ for any type that she reports. Recall that we assume that mechanism Γ^c is IR in the setting $QL(\Gamma^c, u)$. Thus, we assume that

$$q_i(s)v_i(s) + \tau_i^{q_i(s)} \geq \omega^c v_i(s).$$

Expression (20) from Case 2 shows that

$$q_i(s)v_i(s) + \tau_i^{q_i(s)} \geq \omega^c v_i(s) \iff u_i^{1-\omega^c}(\tau_i^{1-\omega^c}(s_{-i}), s) \geq u_i^{\omega^c}(0, s).$$

Thus, we see that if we assume Γ^c is IR in setting $QL(\Gamma^c, u)$ (i.e. the left most inequality is satisfied) implies that mechanism Γ^c satisfies IR in the non-quasilinear setting (i.e. the right most inequality is satisfied) when $s = (s_i, s_{-i}) \in S^n$ and $s_{-i} \in S^{-i}$ is such that $q_i(\tilde{s}_i, s_{-i}) = 1 - \omega^c \ \forall \tilde{s}_i \in S_i$.

Case 6: Suppose that $s = (s_i, s_{-i}) \in S^n$ is such that $q_i(s) = 1 - \omega^c$ and that there exists a $s_i' \in S_i$ such that $q_i(s_i', s_{-i}) = \omega^c$. Thus, we consider the case where $s_{-i} \in S^{-i}$ is such that bidder i is able to influence the number of units that she receives with her report, in contrast to Case 5.

Recall we assume that mechanism Γ^c is EPIC in the non-quasilinear setting. Therefore,

$$u_i^{1-\omega^c}(\tau_i^{1-\omega^c}(s_{-i}), s) \geq u_i^{\omega^c}(\tau_i^{\omega^c}(s_{-i}), s). \tag{21}$$

where the LHS is bidder i's payoff when she truthfully reports her type and the RHS is bidder i's payoff when she misreports type to be $s_i' \in S_i$ and wins $q_i(s_i', s_{-i}) = \omega^c$ units.

We show in Case 4 that $\tau_i^{\omega^c}(s_{-i}) \geq 0$, because we assume IR holds in setting $QL(\Gamma^c, u)$.

When we combine this with Inequality (21) we see that

$$\tau_i^{\omega^c}(s_{-i}) \geq 0 \implies u_i^{1-\omega^c}(\tau_i^{1-\omega^c}(s_{-i}), s) \geq u_i^{\omega^c}(\tau_i^{\omega^c}(s_{-i}), s) \geq u_i^{\omega^c}(0, s).$$

Or equivalently, by substituting $q_i(s) = 1 - \omega^c$, we have that

$$u_i^{q_i(s)}(\tau_i^{q_i(s)}(s_{-i}), s) \geq u_i^{\omega^c}(0, s).$$

Thus, mechanism Γ^c satisfies IR in the non-quasilinear setting in this case.

Cases 4, 5 and 6 combine to show that mechanism Γ^c satisfies IR in the non-quasilinear setting for all $s \in S^n$, when we assume mechanism Γ^c is EPIC in the non-quasilinear setting and IR in setting $QL(\Gamma^c, u)$.

Proof of Theorem 1

If $\Gamma^{\mathbb{S}}$ satisfies EPIC and IR in setting $QL(\Gamma^{\mathbb{S}}, u)$, then Lemmas 2 and 3 shows that $\Gamma^{\mathbb{S}}$ satisfies EPIC and IR in the non-quasilinear setting. Thus, we need to show that if $\Gamma^{\mathbb{S}}$ satisfies EPIC, IR, and efficiency in setting $QL(\Gamma^{\mathbb{S}}, u)$, then $\Gamma^{\mathbb{S}}$ satisfies efficiency in the non-quasilinear setting.

We assume $\Gamma^{\mathbb{S}}$ is efficient in setting $QL(\Gamma^{\mathbb{S}}, u)$. Thus,

$$v_i(s) \geq v_j(s) \text{ if } q_i(s) = 1 \text{ and } j \neq i.$$

By construction of v_i, this implies that

$$d_i^1(\tau_i^0(s_{-i}), s) = v_i(s) \geq v_j(s) = d_j^1(\tau_j^0(s_{-j}), s). \tag{22}$$

We use the above inequality to prove the theorem. To do this, we argue that bidder i's willingness to sell conditional on winning, $d_i^0(\tau_i^1(s_{-i}), s)$, weakly exceeds the left hand side of the above expression. Thus, there are no ex post Pareto improving trades between bidder i and one of her rivals, because bidder i's willingness to sell her unit after the auction $d_i^0(\tau_i^1(s_{-i}), s)$ weakly exceeds any bidder $j's$ willingness to pay for a unit after the auction, which is $d_j^1(\tau_j^0(s_{-i}), s)$.

In order to show this, note that EPIC implies that bidder i best responds by reporting her true type. Thus, bidder i is better off under the outcome where she wins one unit and receives transfer $\tau_i^1(s_{-i})$ versus the outcome where she wins no unit and receives transfer $\tau_i^0(s_{-i})$. This means her willingness to pay for a unit at transfer $\tau_i^0(s_{-i})$ exceeds the difference in transfers,

$$d_i^1(\tau_i^0(s_{-i}), s) \geq \tau_i^0(s_{-i}) - \tau_i^1(s_{-i}) \text{ if } q_i(s) = 1. \tag{23}$$

Note that the above inequality holds by IR if $q_i(\tilde{s}_i, s_{-i}) = 1 \ \forall \tilde{s}_i \in S_i$ as well, because we assume the mechanism is IR and our definition of the corresponding quasilinear setting in this case assumes $\tau_i^0(s_{-i}) = 0$.

If we let $d^* = d_i^1(\tau_i^0(s_{-i}), s)$, then our definitions of d_i^0 and d_i^1 imply that $d_i^0(\tau_i^0(s_{-i}) - d^*, s) = d_i^1(\tau_i^0(s_{-i}), s)$. Thus,

$$d_i^0(\tau_i^1(s_{-i}), s) \geq d_i^0(\tau_i^0(s_{-i}) - d^*, s) = d_i^1(\tau_i^0(s_{-i}), s) \tag{24}$$

where the inequality holds because weakly positive wealth effects imply that d_i^0 is increasing in its first argument, and Expression (23) shows that $\tau_i^1(s_{-i}) \geq \tau_i^0(s_{-i}) - d^*$. Thus,

$$d_i^0(\tau_i^1(s_{-i}), s) \geq d_i^0(\tau_i^0(s_{-i}) - d^*, s) = d_i^1(\tau_i^0(s_{-i}), s) \geq d_j^1(\tau_j^0(s_{-j}), s) \text{ if } q_i(s) = 1 \text{ and } j \neq i,$$

where the first inequality and equality restate Expression (24) and the final inequality follows from (22). Therefore, we have shown that the winning bidder's willingness to sell exceeds her rivals' willingness to pays for any $s \in S^n$. Hence, there are no ex post Pareto improving trades between the winning bidder and her rivals, and $\Gamma^{\mathbb{S}}$ is efficient.

Proof of Lemma 4

For ease of notation, we let $c_i : S^n \to \mathbb{R}_+$ be such that

$$c_i(s) := d_i^0(0, s).$$

The proof is by contradiction. Suppose $\Gamma^{\mathbb{P}}$ satisfies (1)–(4) yet there is a $s \in S^n$ where $q_i(s) = 0$ and there is a bidder $j \neq i$ such that $c_j(s) < c_i(s)$. Thus, we assume there is a procurement mechanism that satisfies our four desired properties, yet the mechanism does not select the lowest cost bidder to be the winning supplier. Since $\Gamma^{\mathbb{P}}$ satisfies IR, then we know that if bidder i wins the procurement auction, she is paid some amount p where $p \geq c_i(s)$. Moreover weakly positive wealth effects imply that $d_i^1(p, s)$ is weakly increasing in p, and the construction of d_i^1 and c_i give that

$$d_i^1(d_i^0(0, s), s) = d_i^1(c_i(s), s) = d_i^0(0, s) = c_i(s).$$

Thus,

$$p \geq c_i(s) \implies d_i^1(p, s) \geq c_i(s).$$

Thus, after the outcome of mechanism $\Gamma^{\mathbb{P}}$, bidder i is made weakly better off by repurchasing a unit of the good from one of her rivals for price $c_i(s)$, because $c_i(s)$ is weakly below her willingness to pay for a unit after the outcome of the mechanism. The lowest cost rival, bidder j, is made strictly better off by selling her unit to bidder i and being paid $c_i(s)$, because her reservation cost of supplying a unit $c_j(s)$ is strictly smaller than $c_i(s)$ by assumption. Hence there is an ex post Pareto improving trade where bidder i pays bidder j the amount $c_i(s)$ to buy her unit. This contradicts efficiency. Thus, any mechanism that satisfies Properties (1)-(4) necessarily selects the lowest cost bidder to be the winning supplier.

Proof of Lemma 5

In this proof, we show that there exists a mechanism that satisfies properties (1)-(4) if and only if a $\Gamma^{\mathbb{P}*}$ mechanism is EPIC and efficient. Note, that it is trivial that there is a mechanism that satisfies properties (1)-(4) if a $\Gamma^{\mathbb{P}*}$ mechanism is EPIC and efficient, because a $\Gamma^{\mathbb{P}*}$ mechanism satisfies IR and no subsidies by construction. Thus, we show the non-trivial direction of the proof below. In particular, we show that there exists a mechanism that satisfies properties (1)-(4) only if a $\Gamma^{\mathbb{P}*}$ mechanism is efficient and EPIC.

In order to show this, we assume there is a mechanism $\tilde{\Gamma}^{\mathbb{P}} = \{\tilde{q}, \tilde{\tau}\}$ that satisfies properties (1)-(4). We assume that mechanism $\tilde{\Gamma}^{\mathbb{P}}$ is not a $\Gamma^{\mathbb{P}*}$ mechanism, because otherwise there obviously is a $\Gamma^{\mathbb{P}*}$ mechanism that satisfies the four properties.

Our proof proceeds in three steps. In Step 1, we specify a particular $\Gamma^{\mathbb{P}*}$ mechanism that we study (note that there are many $\Gamma^{\mathbb{P}*}$ mechanisms, in a technical sense, because we do not specify a tie breaking rule when for the case where there are multiple lowest cost bidders). We call this mechanism $\tilde{\Gamma}^{\mathbb{P}*}$. We then proceed to show that this mechanism $\tilde{\Gamma}^{\mathbb{P}*}$ satisfies EPIC (Step 2), and then we show separately that it satisfies efficiency (Step 3).

Step 1: Note that Lemma 4 shows that the allocation rule of mechanism $\tilde{\Gamma}^{\mathbb{P}}$ always selects the bidder with the lowest cost to be the winning supplier, because $\tilde{\Gamma}^{\mathbb{P}}$ is efficient. Let mechanism $\tilde{\Gamma}^{\mathbb{P}*}$ be the $\Gamma^{\mathbb{P}*}$ mechanism with the same allocation rule as mechanism $\tilde{\Gamma}^{\mathbb{P}}$. There is one $\Gamma^{\mathbb{P}*}$ mechanism with the same allocation rule as mechanism $\tilde{\Gamma}^{\mathbb{P}}$, because we only require that a $\Gamma^{\mathbb{P}*}$ mechanism's allocation rule selects the lowest cost bidder and the allocation rule of mechanism $\tilde{\Gamma}^{\mathbb{P}}$ does. In other words, both mechanisms select the lowest cost bidder to be the winning supplier when there is a unique lowest cost bidder, and both mechanisms use the same tie breaking rule when there is a tie for the lowest cost bidder. Thus, mechanism $\tilde{\Gamma}^{\mathbb{P}}$ and mechanism $\tilde{\Gamma}^{\mathbb{P}*}$ have the same allocation rule that we call $\tilde{q}$. Steps 2 and 3 show that mechanism $\tilde{\Gamma}^{\mathbb{P}*}$ satisfies EPIC and efficiency, respectively. Thus, we show that whenever there is a mechanism $\tilde{\Gamma}^{\mathbb{P}}$ that satisfies EPIC and efficiency, then there is a $\Gamma^{\mathbb{P}*}$ mechanism that is EPIC and efficient. In particular, we show that mechanism $\tilde{\Gamma}^{\mathbb{P}*}$ defined above satisfies EPIC (Step 2) and efficiency (Step 3).

Step 2: In this step we show that mechanism $\tilde{\Gamma}^{\mathbb{P}*}$ satisfies EPIC if mechanism $\tilde{\Gamma}^{\mathbb{P}}$ satisfies Properties (1)-(4). Let $\tilde{\tau}$ describe the transfer rule of mechanism $\tilde{\Gamma}^{\mathbb{P}}$. Similarly, let τ^* represent the transfer rule of mechanism $\tilde{\Gamma}^{\mathbb{P}*}$. Recall that mechanism $\Gamma^{\mathbb{P}} = \{q, \tau\}$ has transfer rule t, where

$$t_i(s) = \begin{cases} \tau_i^0(s_{-i}) & if \ q_i(s) = 0 \\ \tau_i^1(s_{-i}) & if \ q_i(s) = 1 \end{cases}.$$

Note that $\tau_i^{1*}(s_{-i}) = \tilde{\tau}_i^1(s_{-i}) = 0 \ \forall s_{-i} \in S^{-i}$ by construction. In addition,

$$\tau_i^{0*}(s_{-i}) \leq \tilde{\tau}_i^0(s_{-i}) \ \forall s_{-i} \in S^{-i}. \tag{25}$$

To see why Expression (25) holds, note that

$$u_i^0(\tilde{\tau}_i^0(s_{-i}), (s_i, s_{-i})) \geq u_i^1(0, (s_i, s_{-i})) \iff \tilde{\tau}_i^0(s_{-i}) \geq d_i^0(0, s)$$

for all $(s_i, s_{-i}) \in S^n$ such that $\tilde{q}_i(s_i, s_{-i}) = 0$, because we assume mechanism $\tilde{\Gamma}^{\mathbb{P}}$ is IR. Let $\mathcal{S}_i(s_{-i}) = \{s_i | \tilde{q}_i(s_i, s_{-i}) = 0\}$. Then

$$\tilde{\tau}_i{}^0(s_{-i}) \geq \sup_{s_i \in \mathcal{S}_i(s_{-i})} d_i^0(0, (s_i, s_{-i})),$$

because $\tilde{\Gamma}^{\mathbb{P}}$ is IR. The right hand side of the above inequality is equal to $\tau_i^{0*}(s_{-i})$ by construction. Thus, Expression (25) holds.

We argue that $\tilde{\Gamma}^{\mathbb{P}*}$ satisfies EPIC by showing that

$$u_i^{\tilde{q}_i(s_i,s_{-i})}(\tau^{\tilde{q}_i(s_i,s_{-i})*}(s_{-i}), (s_i, s_{-i})) \geq u_i^{\tilde{q}_i(s_i',s_{-i})}(\tau^{\tilde{q}_i(s_i',s_{-i})*}(s_{-i}), (s_i, s_{-i})), \tag{26}$$

for all $s_i, s_i' \in S_i$ and $s_{-i} \in S^{-i}$. In words, the Expression (26) above states that bidder i best responds by truthfully reporting her type given that her rivals report truthfully. We prove that Expression (26) holds by considering three cases.

(a) If $s_i, s_i' \in S_i$ and $s_{-i} \in S^{-i}$ are such that $\tilde{q}_i(s_i, s_{-i}) = \tilde{q}_i(s_i', s_{-i})$, then $\tau^{\tilde{q}_i(s_i,s_{-i})*}(s_{-i}) = \tau^{\tilde{q}_i(s_i',s_{-i})*}(s_{-i})$. Thus,

$$u_i^{\tilde{q}_i(s_i,s_{-i})}(\tau^{\tilde{q}_i(s_i,s_{-i})*}(s_{-i}), (s_i, s_{-i})) = u_i^{\tilde{q}_i(s_i',s_{-i})}(\tau^{\tilde{q}_i(s_i',s_{-i})*}(s_{-i}), (s_i, s_{-i}))$$

(b) If $s_i, s_i' \in S_i$ and $s_{-i} \in S^{-i}$ are such that $\tilde{q}_i(s_i, s_{-i}) = 0$ and $\tilde{q}_i(s_i', s_{-i}) = 1$. Then,

$$u_i^0(\tau_i^{0*}(s_{-i}), (s_i, s_{-i})) \geq u_i^0(d_i^0(0, (s_i, s_{-i})), (s_i, s_{-i})) = u_i^1(0, (s_i, s_{-i}))$$

The first inequality holds because our definition of $\tau_i^{0*}(s_{-i})$ states that $\tau_i^{0*}(s_{-i}) \geq d_i^0(0, (s_i, s_{-i}))$ for all $s_i \in \mathcal{S}_i(s_{-i})$. The second equality holds from our definition of d_i^0. Thus,

$$u_i^{\tilde{q}_i(s_i,s_{-i})}(\tau^{\tilde{q}_i(s_i,s_{-i})*}(s_{-i}), (s_i, s_{-i})) \geq u_i^{\tilde{q}_i(s_i',s_{-i})}(\tau^{\tilde{q}_i(s_i',s_{-i})*}(s_{-i}), (s_i, s_{-i})).$$

(c) If $s_i, s_i' \in S_i$ and $s_{-i} \in S^{-i}$ are such that $\tilde{q}_i(s_i, s_{-i}) = 1$ and $\tilde{q}_i(s_i', s_{-i}) = 0$. Then,

$$u_i^1(0, (s_i, s_{-i})) \geq u_i^0(\tilde{\tau}_i^0(s_{-i}), (s_i, s_{-i})) \geq u_i^0(\tau_i^{0*}(s_{-i}), (s_i, s_{-i})),$$

where the first inequality holds because we assume mechanism $\tilde{\Gamma}^{\mathbb{P}}$ is EPIC and has the same allocation rule; and the second inequality follows because we already showed that $\tilde{\tau}_i^0(s_{-i}) \geq \tau_i^{0*}(s_{-i})$. Thus,

$$u_i^{\tilde{q}_i(s_i,s_{-i})}(\tau^{\tilde{q}_i(s_i,s_{-i})*}(s_{-i}), (s_i, s_{-i})) \geq u_i^{\tilde{q}_i(s_i',s_{-i})}(\tau^{\tilde{q}_i(s_i',s_{-i})*}(s_{-i}), (s_i, s_{-i})).$$

Step 3: In this step we show that mechanism $\tilde{\Gamma}^{\mathbb{P}*}$ satisfies efficiency if mechanism $\tilde{\Gamma}^{\mathbb{P}}$ satisfies Properties (1)-(4).

We assume mechanism $\tilde{\Gamma}^{\mathbb{P}}$ is efficient, and thus there are no ex post Pareto improving trades among bidders. Thus, if bidder i is the winning supply and sells her unit, her willingness to pay for a unit is weakly below her rivals' willingness to sell

$$q_i(s_i, s_{-i}) = 0 \implies \min_{j \neq i} d_j^0(0, (s_i, s_{-i})) \geq d_i^1(\tilde{\tau}_i^0(s_{-i}), (s_i, s_{-i}))$$

Weakly positive wealth effects imply that $d_i^1(\tilde{\tau}_i^0(s_{-i}), (s_i, s_{-i})) \geq d_i^1(\tau_i^{0*}(s_{-i}), (s_i, s_{-i}))$ because in Step 2 above, we show that $\tilde{\tau}_i{}^0(s_{-i}) \geq \tau_i^{0*}(s_{-i})$. Thus,

$$q_i(s_i, s_{-i}) = 0 \implies \min_{j \neq i} d_j^0(0, (s_i, s_{-i})) \geq d_i^1(\tilde{\tau}_i^0(s_{-i}), (s_i, s_{-i})) \geq d_i^1(\tau_i^{0*}(s_{-i}), (s_i, s_{-i})),$$

and hence there are no ex post Pareto improving trades following the outcome mechanism $\tilde{\Gamma}^{\mathbb{P}*}$, making it efficient.

Note that the above proof follows from the weakly positive wealth effects assumption. Specifically, the winning supplier is paid weakly less in mechanism $\tilde{\Gamma}^{\mathbb{P}*}$ relative to mechanism $\tilde{\Gamma}^{\mathbb{P}}$. Thus, the winning supplier has less demand for owning a unit following mechanism $\tilde{\Gamma}^{\mathbb{P}*}$ relative to mechanism $\tilde{\Gamma}^{\mathbb{P}}$, because she is relatively poorer after participating in the former mechanism. Thus, if there are no ex post Pareto improving trades in the latter mechanism, there are no ex post Pareto improving trades in the former — the only difference between the two is that the bidder without a unit has less desire for a unit in the former.

Proof of Theorem 2

Recall Lemma 5 shows that a mechanism satisfies properties (1)-(4) if and only if a $\Gamma^{\mathbb{P}*}$ mechanism satisfies properties (1)-(4). This simplifies our proof because it allows us to to consider only $\Gamma^{\mathbb{P}*}$ mechanisms.

Our proof proceeds in 5 steps. Step 1 shows that $\Gamma^{\mathbb{P}*}$ mechanisms satisfy EPIC for any $\alpha \in [0,1]$. Thus, a $\Gamma^{\mathbb{P}*}$ mechanism satisfies (1) EPIC, (2) IR, and (3) no subsidies for any $\alpha \in [0,1]$.

The remainder of the proof studies when a $\Gamma^{\mathbb{P}*}$ mechanism is efficient. We let $E \subset [0,1]$ be the set of all α where a $\Gamma^{\mathbb{P}*}$ mechanism is efficient. Our goal is to show that $E = [0, \alpha^*]$ where $\alpha^* \in [0,1)$. In other words, we want to show that there is an $\alpha^* \in [0,1)$ such that a $\Gamma^{\mathbb{P}*}$ mechanism satisfies property (4) - efficiency - if and only if the degree of interdependence between bidders is $\alpha \leq \alpha^*$.

Step 2 shows that a $\Gamma^{\mathbb{P}*}$ mechanism is efficient when $\alpha = 0$ (i.e. $0 \in E$). Step 3 shows that a $\Gamma^{\mathbb{P}*}$ mechanism violates efficiency when $\alpha = 1$ (i.e. $1 \notin E$). Step 4 then shows that if a $\Gamma^{\mathbb{P}*}$ mechanism is efficient when the degree of interdependence is $\alpha^h \in (0,1)$, then a $\Gamma^{\mathbb{P}*}$ mechanism is efficient in a setting where the degree of interdependence is $\alpha^\ell \leq \alpha^h$. Let $\alpha^* = \sup_{\alpha \in E} \alpha$. Thus, Step 4 shows that $\alpha \in E$ if $\alpha < \alpha^*$. That is, a $\Gamma^{\mathbb{P}*}$ mechanism is efficient when $\alpha < \alpha^*$. We conclude the proof in Step 5 by showing a $\Gamma^{\mathbb{P}*}$ mechanism is efficient when the degree of interdependence $\alpha = \alpha^*$ (i.e. $\alpha^* \in E$). Thus, Step 2-5 combine to show that $E = [0, \alpha^*]$. When combined with Step 1, this shows that there is a mechanism that satisfies properties (1)-(4) if and only if $\alpha \in [0, \alpha^*] = E$.

We use the notation $c(s_i + \alpha \sum_{j \neq i} s_j) := d_i^0(0, s)$ through the proof.

Step 1: We show that a $\Gamma^{\mathbb{P}*}$ mechanism is EPIC for any $\alpha \in [0,1]$. Consider a $\Gamma^{\mathbb{P}*}$ mechanism. Fix $i \in \{1, \ldots, n\}$ and consider two cases. First, suppose that $s \in S^n$ is such that $q_i(s) = 0$. Definition 8 then implies $s_i \geq \max_{j \neq i} s_j$ and $t_i(s) = \tau_i^{0*}(s_{-i}) = c((\max_{j \neq i} s_j) + \alpha \sum_{j \neq i} s_j)$. Note that

$$c((\max_{j \neq i} s_j) + \alpha \sum_{j \neq i} s_j) \geq c(s_i + \alpha \sum_{j \neq i} s_j),$$

and thus bidder i is paid weakly more than her cost of supplying. Thus, bidder i best responds by truthfully reporting her private signal. If bidder i reports signal $s_i' \neq s_i$ where $q_i(s_i', s_{-i}) = 0$, then her payoff is unchanged; and if bidder i reports $s_i' \neq s_i$ where $q_i(s_i', s_{-i}) = 1$, then she retains her unit and receives no payment. The latter outcome makes bidder i weakly worse off because she sells her unit for amount that weakly exceeds her reservation cost when she truthfully reports her type.

Second, suppose that $s \in S^n$ is such that $q_i(s) = 1$. Thus, bidder i retains her unit and receives no money if she reports truthfully. Definition 8 then implies $s_i \leq \max_{j \neq i} s_j$ and $t_i(s) = \tau_i^{0*}(s_{-i}) = c((\max_{j \neq i} s_j) + \alpha \sum_{j \neq i} s_j)$. Note that

$$c((\max_{j \neq i} s_j) + \alpha \sum_{j \neq i} s_j) \leq c(s_i + \alpha \sum_{j \neq i} s_j),$$

because $s_i \leq \max_{j \neq i} s_j$. Thus, bidder i best responds by truthfully reporting her private signal. If bidder i reports signal $s_i' \neq s_i$ where $q_i(s_i', s_{-i}) = 1$, then her payoff is unchanged; and if bidder i reports $s_i' \neq s_i$ where $q_i(s_i', s_{-i}) = 0$, then she is the winning supplier, but she is paid an amount $\tau_i^{0*}(s_{-i})$ that is weakly below her reservation cost of supplying her unit. Thus, she is made no better off by misreporting her type.

Step 2: Next, we show that a $\Gamma^{\mathbb{P}*}$ mechanism is efficient if $\alpha = 0$. First, note that if $x > y$, then

$$d_i^1(y, s) + x - y > d_i^1(x, s), \tag{27}$$

This follows directly from our definition of d_i^1, because

$$x > y \implies u_i^1(x - d_i^1(x, s), s) = u_i^0(x, s) > u_i^0(y, s) = u_i^1(y - d_i^1(y, s), s),$$

and

$$u_i^1(x - d_i^1(x, s), s) > u_i^1(y - d_i^1(y, s), s) \implies x - d_i^1(x, s) > y - d_i^1(y, s) \implies d_i^1(y, s) + x - y > d_i^1(x, s).$$

Without loss of generality, suppose that bidder 1 is the winning bidder and bidder 2 is her lowest cost rival. Thus, $s_1 \geq s_2 \geq s_j \ \forall j \in \{3, \ldots, n\}$. Note that $\tau_1^{0*}(s_{-1}) = c(s_2)$ when $\alpha = 0$. Thus, bidder 1 is willing to pay $d_1^1(c(s_2), s)$ for a unit of the good after she supplies the unit she is endowed with and paid $\tau_1^{0*}(s_{-1})$. Bidder 2 is the rival with the lowest reservation cost of supplying bidder 1 with a unit. Bidder 2's reservation cost is $c(s_2)$. The $\Gamma^{\mathbb{P}*}$ mechanism's outcome is efficient if bidder 1's willingness to pay for a unit (after being paid to supply her unit to the auctioneer) is below bidder 2's reservation cost of supplying her unit:

$$d_1^1(c(s_2), s) \leq c(s_2) \ \forall s \in S^n \ s.t. \ s_1 \leq s_2 \leq s_j \ \forall j \in \{3, \ldots, n\}. \tag{28}$$

If $s_1 = s_2$, then the above expression holds with equality because the definition of d_1^1 implies

$$d_1^1(c(s_1), s) = c(s_1) = c(s_2).$$

If $s_1 > s_2$, then Expression (27) implies

$$d_1^1(c(s_2), s) < d_1^1(c(s_1), s) + c(s_2) - c(s_1) = c(s_2),$$

where the equality holds because $d_1^1(c(s_1), s) = c(s_1)$. Thus, the outcome of mechanism $\Gamma^{\mathbb{P}*}$ is efficient, because Expression (28) holds.

Step 3: Next we show that all $\Gamma^{\mathbb{P}*}$ mechanisms are inefficient when α is sufficiently close to one. We show the mechanisms are inefficient by showing that there are signal realizations under which there are ex post Pareto improving trades between the winning bidder and her lowest cost rival for the case where α is sufficiently close to one.

Fix $s \in [0,1]^n$ and suppose that s is such that

$$s_1 > s_2 \geq s_j \ \forall j \in \{3, \ldots, n\}.$$

Thus, all $\Gamma^{\mathbb{P}*}$ mechanisms select bidder 1 to be the winning supplier. Bidder 1 is paid $c(s_2 + \alpha \sum_{j \neq 1} s_j)$ to supply the unit. We show that there is an ex post Pareto improving trade where bidder 1 buys a unit from bidder 2 following mechanism $\Gamma^{\mathbb{P}*}$ if α is close to one. There is an ex post Pareto improving trade if for some $s \in S^n$ such that $s_1 > s_2 \geq s_j \ \forall j \in \{3, \ldots, n\}$,

$$d_1^1(c(s_2 + \alpha \sum_{j \neq 1} s_j), s) > c(s_2 + \alpha \sum_{j \neq 2} s_j).$$

In words, the winning supplier's willingness to pay to buy a unit (after selling her unit to the auctioneer) exceeds her lowest cost rival's cost of supplying a unit. To condense notation let $f(s, \alpha) = d_1^1(c(s_2 + \alpha \sum_{j \neq 1} s_j), s)$ be bidder 1's willingness to pay for a unit conditional upon being the winning supplier. In addition, let $g(s, \alpha) = c(s_1 + \alpha \sum_{j \neq 1} s_j)$ be bidder 1's reservation cost of supplying a unit, and let $h(s, \alpha) = c(s_2 + \alpha \sum_{j \neq 2} s_j)$ be bidder 2's reservation cost for supplying a unit. Note that f, g, and h are all continuous in α. In addition,

$$f(s, \alpha) = d_1^1(c(s_2 + \alpha \sum_{j \neq 1} s_j), s) > c(s_1 + \alpha \sum_{j \neq 1} s_j) = g(s, \alpha),$$

because

$$d_1^1(c(s_2 + \alpha \sum_{j \neq 1} s_j), s) > d_1^1(c(s_1 + \alpha \sum_{j \neq 1} s_j), s) = c(s_1 + \alpha \sum_{j \neq 1} s_j) \tag{29}$$

where the inequality follows because bidder 1 is paid above her reservation cost and she has strictly positive wealth effects, and the equality follows from the definition of d_1^1. Furthermore, note that $g(s, 1) = h(s, 1)$. Thus,

$$f(s, \alpha) > g(s, \alpha) \; \forall \alpha \in [0, 1] \implies f(s, 1) - g(s, 1) = f(s, 1) - h(s, 1) > 0.$$

Since both f and h are continuous in α, we can then say that for a fixed $s \in S^n$ (where $s \in s_1 > s_2 \geq s_j \; \forall j \in \{3, \ldots, n\}$) there exists an $\tilde{\alpha} < 1$ such that

$$f(s, \alpha) - h(s, \alpha) > 0 \; \forall \alpha \in [\tilde{\alpha}, 1].$$

In words, this implies that there is an ex post Pareto improving trade in any $\Gamma^{\mathbb{P}*}$ mechanism when $s \in S^n$ and α is sufficiently close to one. The ex post Pareto improving trade between bidder 1 and bidder 2 is for price $p \in (h(s, \alpha), f(s, \alpha))$, where $f(s, \alpha) > h(s, \alpha)$ for α close to 1.

Step 4: We show that if a $\Gamma^{\mathbb{P}*}$ mechanism is efficient when the degree of interdependence is $\alpha^h \in (0, 1)$, then any $\Gamma^{\mathbb{P}*}$ mechanism is efficient in a setting where the degree of interdependence is $\alpha^\ell \leq \alpha^h$.

For ease of notation, we write bidder i's willingness to pay for the good conditional on receiving transfers t as

$$\tilde{d}_i^1(t, s_i + \alpha \sum_{j \neq i} s_j) := d_i^1(t, s) \; \forall t \in \mathbb{R}, \; s \in [0, 1]^n.$$

Note that we can condense the willingness to pay function to the $\tilde{d}_i^1$ form because of our assumptions on U^0 and U^1.

Consider the case where $s \in [0, 1]^n$ is such that $s_i > \max_{j \neq i} s_j$ (We study the edge-case where $s \in [0, 1]^n$ is such that $s_i = \max_{j \neq i} s_j$ at the end of this step). Without loss of generality suppose that $s_1 > s_2 \geq s_j \; \forall j \in \{3, \ldots, n\}$. Thus, bidder 1 is selected as the winning bidder in any $\Gamma^{\mathbb{P}*}$ mechanism. A $\Gamma^{\mathbb{P}*}$ mechanism is efficient if there are no ex post Pareto improving trades between the winning bidder — who we call bidder 1 — and her lowest cost rival — who we call bidder 2. Thus, we want to show that when the degree of interdependence in bidder preferences is α^ℓ,

$$\tilde{d}_1^1(c(s_2 + \alpha^\ell \sum_{j \neq 1} s_j), s_1 + \alpha^\ell \sum_{j \neq 1} s_j) \leq c(s_2 + \alpha^\ell \sum_{j \neq 2} s_j), \tag{30}$$

because the left hand side is bidder 1's willingness to pay conditional on being the winning supplier, and the right hand side is bidder 2's cost of supplying a unit.

Note, that we assume that there is a $\Gamma^{\mathbb{P}*}$ mechanism that is efficient in a setting where the level of interdependence is $\alpha^h \geq \alpha^\ell$. Thus, we assume that for any $s' \in [0,1]^n$ such that $s'_1 > s'_2 \geq s'_j \; \forall j \in \{3, \ldots, n\}$, then

$$\tilde{d}_1^1(c(s'_2 + \alpha^h \sum_{j \neq 1} s'_j), s'_1 + \alpha^h \sum_{j \neq 1} s'_j) \leq c(s'_2 + \alpha^h \sum_{j \neq 2} s'_j). \tag{31}$$

The above inequality holds because efficiency implies there are not ex post Pareto improving trades following the outcome of the $\Gamma^{\mathbb{P}*}$ mechanism. Thus, it is the case that the winning supplier's willingness to pay to repurchase a unit (after being paid $\tau_i^{0*}(s_{-i})$ to supply her unit) is below her lowest cost rivals cost of supplying a unit — because if the reverse inequality held, there would be an ex post Pareto improving trade between bidders 1 and 2.

We complete the proof by using Expression (31) to prove that Expression (30) holds. In particular, we show that for any $s \in [0,1]^n$ such that $s_1 > s_2 \geq s_j$ for all $j \in \{3, \ldots, n\}$, that there is a $\tilde{s} \in [0,1]^n$ such that $\tilde{s}_1 > \tilde{s}_2 \geq \tilde{s}_j$ for all $j \in \{3, \ldots, n\}$ where

$$\begin{aligned} s_2 + \alpha^\ell \textstyle\sum_{j \neq 2} s_j &= \tilde{s}_2 + \alpha^h \textstyle\sum_{j \neq 2} \tilde{s}_j, \\ s_1 + \alpha^\ell \textstyle\sum_{j \neq 1} s_j &= \tilde{s}_1 + \alpha^h \textstyle\sum_{j \neq 1} \tilde{s}_j, \\ s_2 + \alpha^\ell \textstyle\sum_{j \neq 1} s_j &= \tilde{s}_2 + \alpha^h \textstyle\sum_{j \neq 1} \tilde{s}_j. \end{aligned} \tag{32}$$

Note, the system of equations defined above is solved by setting

$$\begin{aligned} \tilde{s}_1 &= s_1 + s_2(\alpha^\ell - \alpha^h \tfrac{1+\alpha^\ell}{1+\alpha^h}), \\ \tilde{s}_2 &= \tfrac{1+\alpha^\ell}{1+\alpha^h} s_2, \\ \tilde{s}_j &= \tfrac{\alpha^\ell}{\alpha^h} s_j. \end{aligned}$$

The construction of $(\tilde{s}_1, \ldots, \tilde{s}_n)$ is such that $\tilde{s}_j \in [0,1]$ and $\tilde{s}_j \leq s_j \; \forall j = 1, \ldots, n$. Both follow immediately from the construction of $\tilde{s}_j$ when $j = 2, \ldots, n$. When $j = 1$, note that $-1 \leq \alpha^\ell - \alpha^h \frac{1+\alpha^\ell}{1+\alpha^h} \leq 0$ because $0 \leq \alpha^\ell \leq \alpha^h \leq 1$. Hence, $s_2 < s_1 \implies 0 \leq \tilde{s}_1 \leq s_1$.

Moreover, $\tilde{s}_1 > \tilde{s}_2 \geq \tilde{s}_j$, where $j \neq 1, 2$. To see that $\tilde{s}_1 \geq \tilde{s}_2$ note that if it were the case that $s_1 = s_2$, then $\tilde{s}_1 = \tilde{s}_2$. In addition, $\tilde{s}_1$ is increasing in s_1 and $\tilde{s}_2$ is unchanged in s_1. Hence, $s_1 > s_2 \implies \tilde{s}_1 > \tilde{s}_2$. To prove that $\tilde{s}_2 \geq \tilde{s}_j \; \forall j \in \{3, \ldots, n\}$ note that $1 \geq \alpha^h \geq \alpha^\ell \geq 0 \implies \frac{1+\alpha^\ell}{1+\alpha^h} \geq \frac{\alpha^\ell}{\alpha^h}$. Thus

$$\tilde{s}_2 = \frac{1+\alpha^\ell}{1+\alpha^h} s_2 \geq \frac{\alpha^\ell}{\alpha^h} s_j = \tilde{s}_j \; \forall j \neq 1, 2. \tag{33}$$

Therefore, because $\tilde{s} \in [0,1]^n$ is such that $\tilde{s}_1 > \tilde{s}_2 \geq \tilde{s}_j$ for all $j \in \{3, \ldots, n\}$, Expression (31) implies that

$$\tilde{d}_1^1(c(\tilde{s}_2 + \alpha^h \sum_{j \neq 1} \tilde{s}_j), \tilde{s}_1 + \alpha^h \sum_{j \neq 1} \tilde{s}_j) \leq c(\tilde{s}_2 + \alpha^h \sum_{j \neq 2} \tilde{s}_j). \tag{34}$$

Or equivalently, when we substitute using the three equations in Expression (32) we have that

$$\tilde{d}_1^1(c(s_2 + \alpha^\ell \sum_{j \neq 1} s_j), s_1 + \alpha^\ell \sum_{j \neq 1} s_j) \leq c(s_2 + \alpha^\ell \sum_{j \neq 2} s_j), \tag{35}$$

which is what we wanted to show. Thus, there are no ex post Pareto improving trades, and the outcome of a $\Gamma^{\mathbb{P}*}$ mechanism is an efficient outcome if $\alpha^\ell \leq \alpha^h$.

To complete the proof we show that the outcome of mechanism any $\Gamma^{\mathbb{P}*}$ is efficient when the degree of interdependence is α^ℓ and $s \in S^n$ is such that that (1) $s_i = \max_{j \neq i} s_j$ and (2) $q_i(s) = 1$. Thus, we assume there is a tie for the highest signal and without loss of generality we consider a $\Gamma^{\mathbb{P}*}$ mechanism where bidder i is selected as the winner. Note,

$$\tau_i^{0*}(s_{-i}) = c(\max_{j \neq i} s_j + \alpha^\ell \sum_{j \neq i} s_j) = c(s_i + \alpha^\ell \sum_{j \neq i} s_j)$$

where the last equality follows because we assume $s_i = \max_{j \neq i} s_j$. Thus, bidder i is paid her cost to supply her unit. Thus, her willingness to pay for a unit after winning and being paid her reservation cost equals her reservation cost:

$$d_i^1(c(s_i + \alpha^\ell \sum_{j \neq i} s_j), s) = c(s_i + \alpha^\ell \sum_{j \neq i} s_j).$$

In addition $s_i = \max_{j \neq i} s_j$ implies that

$$c_i(s_i + \alpha^\ell \sum_{j \neq i} s_j) \leq c_m(s_m + \alpha^\ell \sum_{j \neq m} s_j) \ \forall m \in \{1, \ldots, i-1, i+1, \ldots, n\}.$$

Thus, the outcome of the $\Gamma^{\mathbb{P}*}$ mechanism is efficient in this case because the winning supplier's willingness to sell is weakly below any of her rivals' willingnesses to pay.

Thus, we have shown that the outcome of any $\Gamma^{\mathbb{P}*}$ mechanism is efficient for any $s \in [0,1]^n$ when the degree of interdependence in bidder preferences is $\alpha^\ell \leq \alpha^h$.

Step 5: Recall that E is the set of all $\alpha \in [0,1]$ where a $\Gamma^{\mathbb{P}*}$ mechanism is efficient and $\alpha^* = \sup_{\alpha \in E} \alpha$. We conclude our proof by showing that $\alpha^* \in E$. In other words, a $\Gamma^{\mathbb{P}*}$ mechanism is efficient if the degree of interdependence is α^*.

Recall that Step 4 implies that a $\Gamma^{\mathbb{P}*}$ mechanism is efficient if the degree of interdependence between bidders is $\alpha < \alpha^*$. In addition, the final part of Step 4 shows that the outcome of a $\Gamma^{\mathbb{P}*}$ mechanism is efficient for any $\alpha \in [0,1]$ when there is a tie for the lowest cost bidder (i.e. if $s_1 = s_2 \geq s_j \ \forall j \in \{3, \ldots, n\}$).

Thus, to complete the proof, we show that a $\Gamma^{\mathbb{P}*}$ mechanism is efficient when the degree of interdependence is α^* and there is a unique lowest cost bidder. Without loss of generality, suppose that $s_1 > s_2 \geq s_j \ \forall j \in \{3, \ldots, n\}$. Thus, bidder 1 is selected as the winning bidder. We know that a $\Gamma^{*\mathbb{P}}$ mechanism is efficient for any setting where $\alpha < \alpha^*$. Equivalently,

$$\tilde{d}_1^1(c(s_1 + \alpha \sum_{j \neq 1} s_j), s_1 + \alpha \sum_{j \neq 1} s_j) \leq c(s_2 + \alpha \sum_{j \neq 2} s_j) \ \forall \alpha < \alpha^*,$$

where the LHS is bidder 1's willingness to pay for a unit and the RHS is her lowest cost rival's willingness to sell. Both the LHS and RHS are continuous in α. Thus,

$$\lim_{\alpha \to^- \alpha^*} \tilde{d}_1^1(c(s_1 + \alpha \sum_{j \neq 1} s_j), s_1 + \alpha \sum_{j \neq 1} s_j) \leq \lim_{\alpha \to^- \alpha^*} c(s_2 + \alpha \sum_{j \neq 2} s_j).$$

In addition, continuity implies we can evaluate each limit above at α^*. Thus,

$$\tilde{d}_1^1(c(s_1 + \alpha^* \sum_{j \neq 1} s_j), s_1 + \alpha^* \sum_{j \neq 1} s_j) \leq c(s_2 + \alpha^* \sum_{j \neq 2} s_j).$$

Thus, a $\Gamma^{\mathbb{P}*}$ mechanism is efficient when the degree of interdependence is α^*, because there is no ex post Pareto improving trade following the mechanism for any $s \in S^n$.

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