Overbidding and Inefficiencies in Multi-unit Vickrey Auctions for Normal Goods

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Abstract

I examine bid behavior in uniform-price auctions and multi-unit Vickrey auctions, without the standard quasilinearity restriction on bidder preferences. Instead of assuming quasilinearity, I assume that bidders have weakly positive wealth effects, i.e. the goods are normal goods. My setting nests quasilinearity, but also allows for budget constraints, financial constraints, and risk aversion. I show that without the quasilinearity restriction, truthful reporting is not a dominant strategy in the Vickrey auction. Instead, bidders truthfully demand for their first unit and weakly overreport their demand for later units. The incentive to overreport demand means that the Vickrey auction is generally inefficient. This mirrors the well-known demand reduction results in uniform-price auctions. Moreover, the efficiency ranking of the two auctions is ambiguous. In fact, the efficiency ranking of the two auctions can reverse, even if only one bidder has non-quasilinear preferences.

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1 Introduction

Governments often hold auctions for the stated goal of allocating resources to the public efficiently. Milton Friedman's 1960 report on treasury auctions suggests that policy makers should use uniform-price auctions to efficiently allocated resources. In the decades that have followed, the uniform-price auction has been adopted to allocate treasury bills, electricity, and government-issued licenses. However, the demand-reduction result of Ausubel et al. (2014) shows that in uniform-price auctions, bidders have a strategic incentive to underreport their demands to the auctioneer. This demand reduction incentive implies that the uniform-price auction is inefficient. In contrast, in the benchmark private value auction setting, truthful reporting demand is a dominant strategy in the Vickrey auction and the auction yields an efficient allocation.

Yet, we rarely see the Vickrey auction implemented in practice. Ausubel and Milgrom (2006) provide four critiques of the Vickrey-Clarke-Groves (VCG) mechanism that suggest why VCG is rarely used in practice. However, their critiques are limited to heterogenous good settings where bidders have complements preferences. On the sale of homogenous goods, Rothkopf (2007) argues that Vickrey auctions are impractical because they are susceptible to collusion and perform poorly when bidders have hard budget constraints.[1] However, the uniform-price auction also performs poorly in the face of budgets, and is similarly susceptible to collusion. Thus, it is unclear why homogenous goods are frequently sold using the uniform-price auction instead of the Vickrey auction.

In this paper, I propose an explanation for why the uniform-price auction is used in favor of the Vickrey auction for the allocation of homogenous goods. I remove the standard quasilinearity restriction on bidder preferences and consider a general preference domain that nests quasilinearity, but also allows risk aversion, financial constraints, and/or budgets. I show that without quasilinearity, the Vickrey auction loses its desirable incentive and efficiency properties. Instead of truthfully reporting their preferences, bidders truthfully report their demand for their first unit and overreport their demand for later units. We see that the Vickrey auction is generally inefficient because bidders have an incentive to misreport their preferences. This result mirrors the demand reduction result of Ausubel et al. (2014). I use my results on bid behavior to show that there is no clear efficiency ranking between the uniform-price auction in the Vickrey auction. In fact, there are cases where the uniform-price auction yields a Pareto efficient allocation of resources, and the Vickrey

[1] Rothkopf (2007) also states Vickrey auctions require bidders to reveal valuable private information. Yet, Ausubel clinching auction (2004), provides a dynamic implementation of the Vickrey auction with desirable privacy preservation properties. In addition, Rothkopf provides other critiques that are relevant only in heterogenous good settings.

outcome is Pareto dominated. This can occur even if only one bidder has non-quasilinear preferences.

Prior work in the empirical auctions literature have compared the performance of Vickrey auctions and uniform-price auctions under the quasilinearity restriction. Most notably, Hortaçsu and Puller (2008) compare the performance of Vickrey auctions and uniform-price auctions in the Texas electricity spot market. They show that when we assume bidder preferences are quasilinear, there is a non-negligible efficiency loss associated with using the uniform-price auction instead of the Vickrey auction. Similarly, Fabra, von der Fehr, and Harbord (2002) study electricity markets and suggest policy makers use Vickrey auctions instead of uniform-price auctions, citing efficiency concerns.

However, the analysis in these papers assumes that bidders have quasilinear preferences. In auction theory, it is common to assume that bidders have quasilinear preferences for tractability. Yet, in many relevant economic environments, bidders are risk averse, have budgets, or face financing constraints.[2] Thus, allowing for non-quasilinear preferences provides a more complete description of bidder preferences.

I show that relaxing quasilinearity can reverse the efficiency ranking of the uniform-price and Vickrey auctions. I consider a setting where bidders have private values and multi-unit demands. By construction, in the Vickrey auction the price a bidder pays to acquire her first unit is lower than the price she pays for her second unit. If a bidder wins two units, then the payment rule is equivalent to having the bidder pay the (relatively higher) price for the second unit for both units, and then refunding her the difference in the two prices. This refund increases a bidders demand due to weakly positive wealth effects. Thus the bidder has an incentive to overreport her demand curve. I formalize this intuition, and show that any bid profile that understates a bidder's demand is weakly dominated.

For uniform-price auctions, I show that the intuition of Ausubel et al. (2014) holds when we remove the quasilinearity restriction. That is, any bid profile that overstates a bidder's demand and/or misreports her demand for her first unit is weakly dominated. However, bidders may have an incentive to underreport demand. In both cases, I form a partial characterization of bid behavior by looking at undominated strategies. While explicitly characterizing bid behavior is intractable, I show that considering only undominated strategies is sufficient for efficiency comparisons between the two auctions.

I use my bounds on bid behavior to develop an example that illustrates the ambiguity of the efficiency ranking the two auctions. In my example, only one bidder has non-quasilinear preferences, yet the outcome of the Vickrey auction is inefficient with positive probability.

[2]See Che and Gale (1998) for examples on financing constraints, and Maskin (2000) for examples on budgets.

But, the outcome of the uniform-price auction is Pareto efficient with probability one.

The rest of the paper proceeds as follows. The remainder of the introduction relates my work to the auctions literature. Section 2 describes my model and provides a brief description of the uniform-price auction and the Vickrey auction. Section 3 proves results on bid behavior in both auctions. Section 4 concludes.

Related Literature

Much of this prior literature about auctions assumes quasilinearity, but there is a literature study auctions with more general preferences. In the single unit environment, Matthews (1983) and Che and Gale (2006) show that risk aversion explains the experimental finding that first price auctions have higher revenues than second price auctions. Maskin and Riley (1984) and Baisa (2015) study the auction design problem when bidders do not have quasilinear preferences. Of this prior work, only Che and Gale (2006) and Baisa (2015) allow multidimensional heterogeneity across risk preferences and wealth effects like the setting studied here. Outside of the auctions literature, Garratt and Pycia (2014) discuss the efficient allocation of a normal good in a bilateral trade setting and show that we get qualitatively different results from those of Myerson and Satterthwaite (1983).

In the multi-unit auctions literature, most work that studies bidders with non-quasilinear preferences looks at the case where bidders have hard budget constraints. Recently, Dobzinski, Lavi and Nisan (2012) showed that when bidders have private budgets, there is no dominant strategy mechanism that implements a Pareto efficient allocation and respects incentive compatibility when transfers are non-positive. In a related paper, Hafalir, Ravi and Sayedi (2012) study a modified Vickrey auctions for divisible goods when bidders have budgets. Like this paper, they bound bid behavior by eliminating dominated strategies. Morimoto and Serizawa (2013) also study efficiency in multi-unit auctions when bidders have non-quasilinear preferences, but in a setting where bidders have single unit demands.

While uniform-price auctions are generally inefficient, the large auctions literature shows that these inefficiencies become negligible when there are many bidders (see Swinkles (1999, 2001), Jackson and Kremer (2006), or Azevedo and Budish (2013)). An earlier draft of this paper studied the large auctions question as well. In the earlier draft, I show that both the uniform-price and Vickrey auctions are approximately efficient when there are many bidders and many objects.

2 Model

2.1 Bidder Preferences

Consider a seller with k indivisible homogenous goods and bidders $1, \dots, N$. Bidder i is characterized by her initial wealth $w^i \in \mathbb{R}$ and her preferences u^i where

$$u^i : \{0, 1, \dots k\} \times \mathbb{R} \to \mathbb{R}.$$

Bidder i gets utility $u^i(x, w)$ when she owns x objects and has wealth w. I assume $u^i(x, \cdot)$ is strictly increasing and continuous for any $x = 0, 1, \dots k$. In addition, the objects being sold at auction are goods. Thus, $u^i(x, w) > u^i(y, w)$ if and only if $x > y$. I make only two additional assumption on bidder preferences.

First, I assume weakly declining demand for additional units. That is, if a bidder is unwilling to pay p for her x^{th} object, then she is unwilling to pay p for her $(x+1)^{th}$ object. This assumption ensures that bidders have downward sloping (inverse) demand curves and generalizes the declining marginal values assumption imposed in the benchmark quasilinear setting.

Assumption 1. (Weakly declining demand)
If $x \in \{1, \dots, k-1\}$ and

$$u^i(x-1, w) \geq u^i(x, w-p),$$

then

$$u^i(x, w) \geq u^i(x+1, w-p).$$

Second, I assume that bidders have weakly positive wealth effects. This means a bidder's demand does not decrease as her wealth increases. To be more concrete, suppose that bidder i was faced with the choice between two bundles of goods. The first bundle has x goods and costs p_x total, and the second bundle has y goods and costs p_y total, where we assume $x > y$. If bidder i weakly prefers the bundle with more goods, then weakly positive wealth effects states that she also weakly prefers the bundle with more goods if her wealth increases. This is a multi-unit generalization of Cook and Graham's (1977) definition of an indivisible, normal good.

Assumption 2. (Weakly positive wealth effects)
Suppose $x > y$ where $x, y \in \{0, 1, \dots k\}$. Bidder i has weakly positive wealth effects if

$$u^i(x, w - p_x) \geq u^i(y, w - p_y) \implies u^i(x, w' - p_x) \geq u^i(y, w' - p_y) \ \forall w' > w.$$

Let $\mathcal{U}$ denote the set of all utility functions u that satisfy Assumptions 1 and 2.

In the Vickrey and uniform-price auctions, bidders compete by submitting k dimensional bids that represent their demands. I define a bidder's (inverse) demand curve following the standard way.

Definition 1. (Inverse demand curve)
Bidder i has an (inverse) demand curve p^i, where

$$p^i(m) := \max\{p : u^i(m, w^i - pm) \geq u^i(m', w^i - pm')\} \; \forall m' \in \{0, 1, \ldots, k\}.$$

The inverse demand for m units $p^i(m)$ is the highest price at which bidder i demands m objects. Note that $p^i(m)$ is weakly decreasing in m and $\infty > p^i(m) > 0 \; \forall m \in \{1, \ldots, k\}$.

In comparison, with quasilinearity bidder i's preferences are described by a vector of her marginal valuations $v^i \in \mathbb{R}^k_+$, where $v^i_1 \geq v^i_2 \geq \ldots \geq v^i_k$. Her utility function is then

$$u^i(x, w) = \sum_{j=1}^{x} v^i_j + w.$$

Thus, my setting nests quasilinearity, because the above function satisfies assumptions 1 and 2.

I do not place any other restrictions on a bidder's preferences beyond assumptions 1 and 2. Thus, my setting allows for multidimensional heterogeneity across bidders, heterogeneity in wealth effects, risk aversion and financial constraints. Budgets can be modeled as a limiting case where a bidder gets high disutility of spending money beyond a certain threshold. Below, I show two examples of preferences included in this framework. In the first example, bidders are financially constrained. In order to finance their expenditures, they must borrow money, and they face an increasing interest rate on additional borrowing. In the second example, the goods are shares of a risky asset. Tomorrow share price is a draw of a random variable with a commonly known distribution of returns. Bidders are risk averse and display decreasing absolute risk aversion. Thus, as bidders become wealthier, they become more risk tolerant and demand more shares of the risky asset.

Example 1. (Financially constrained bidders)

Bidder i receives v^i_m utils for her m^{th} marginal object. Her utility of wealth is additively separable. She finances her payments by selling bonds. She pays a higher interest rate as she borrows more money. Her preferences are described by u^i, where

$$u^i(x, w) = \left(\sum_{j=1}^{x} v^i_j\right) + f^i(w),$$

where f^i is concave and strictly increasing.

Example 2. (Risky assets)

Bidder i is risk averse and maximizes her expected utility over final wealth levels. She receives utility $g^i(w)$ from wealth w, where g^i is increasing and displays decreasing absolute risk aversion. She is considering buying shares of a risky asset. A share of the asset will be worth s tomorrow, where s is the draw of a random variable with density f. Her (expected) utility from owning x shares and having wealth w is,

$$u^i(x,w) = \int_{s\in\mathbb{R}} g^i(w+xs)f(s)ds.$$

Both uniform-price and Vickrey auctions are mechanisms that map bids to feasible outcomes. A (feasible) outcome of the auction describes each bidder's allocation and transfers. The allocation is described by $x \in \{0,1,\ldots,m\}^n$ where x^i is the number of objects won by bidder i. I define X as the set of all $x \in \{0,1,\ldots,m\}^n$ such that $\sum_i^n x^i \leq k$. Transfers are described by $t \in \mathbb{R}^n$, where t^i is the transfer made by bidder i. My description of the two auctions and notation are similar to Krishna (2002).

2.2 The Uniform-price Auction

In the uniform-price auction, each bidder submits a k dimensional bid. The highest k bids (out of the total the $k \times N$ submitted bids) win. Each good is sold at the same market price that equals the highest losing bid. I let β represent the space of all feasible bids. It is a subset of $\mathbb{R}^k_+$, where

$$\beta := \{b \in \mathbb{R}^k_+ | \infty > b_1 \geq b_2 \geq \ldots b_k\}.$$

The collection of bids from all bidders except i is b^{-i}. I let $x^i_U(b^i, b^{-i})$ be the number of objects bidder i wins in the uniform-price auction given bids $(b^i, b^{-i}) \in \beta^N$. Similarly, I let $t^i_U(b^i, b^{-i})$ be the payment made by bidder i.

Let c^{-i} be the vector of competing bids faced by bidder i, ordered from highest to lowest. Bidder i wins exactly $m \in \{0,1,\ldots k\}$ objects, if she submits m bids that rank in the top k.

$$b^i_m > c^{-i}_{k+1-m} \text{ and } c^{-i}_{k-m} > b^i_{m+1}.$$

For simplicity, I assume that all ties are broken in favor of the higher numbered bidder.

If bidder i wins m objects, she pays mp_U, where $p_U = \max\{b^i_{m+1}, c^{-i}_{k+1-m}\}$ is the value of the highest losing bid.

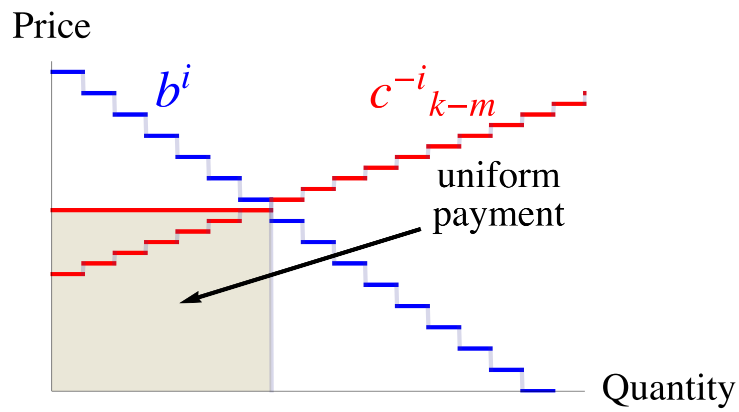

Figure 1: Payment in Uniform-price Auction

2.3 The Vickrey Auction

The Vickrey auction has the same winning rule as the uniform-price auction, however, the payment rule is different. The price a bidder pays for objects is determined by other bidders' reported demands. Specifically, a bidder's payment is determined by a marginal price curve which is a residual demand curve formed by other bidders' reported demand curves.

I keep the same notation as above. Bidder i reports a k dimensional bid $b^i \in \beta$. I let $x_V^i(b^i, b^{-i})$ be the number of objects bidder i wins in the Vickrey auction given bids $(b^i, b^{-i}) \in \beta^N$. Similarly, I let $t_V^i(b^i, b^{-i})$ be the payment made by bidder i.

If bidder i wins m objects, she pays $\sum_{j=1}^{m} c_{k+1-j}^{-i}$. That is, bidder i faces an upward sloping marginal price curve because c_{k+1-j}^{-i} is increasing in j. The marginal price of acquiring the m^{th} object is the $k+1-m^{th}$ highest bid made by i's competitors.

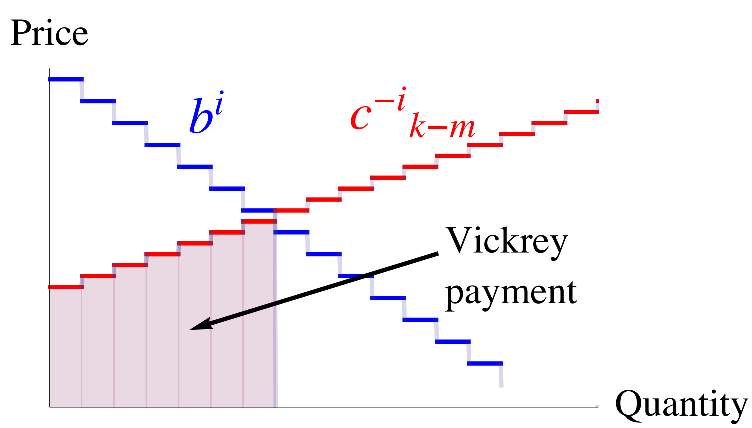

Figure 2: Payment in Vickrey auction

3 Bid Behavior and Inefficiencies

3.1 Uniform-price auctions

Ausubel and Cramton (2002) show that when bidder preferences are quasilinear, there is an incentive to underreport demand in the uniform-price auction. However, bidders will always truthfully report their demand for their first unit of the good. There is a clear intuition for this result. When a bidder desires multiple units, there is a chance that one of her later bids (not the first) will be pivotal in determining her allocation. In particular, the bidder could submit some number of winning bids and also the highest losing bid. In this case, the bidder would like to shade her reported value for later units, because it will reduce the price she pays for her earlier units. I show that this result extends more generally to a setting without the quasilinearity restriction.

In particular, I show overreporting demand is a dominated strategy. In addition, misreporting demand for your first unit is also weakly dominated. This result is summarized in Proposition 1.

Proposition 1. (No overreporting in uniform-price auctions)
The bid $b^i = (b^i_1, b^i_2, \ldots, b^i_k) \in \beta$ is weakly dominated by the bid

$$\tilde{b}^i = (p^i(1), b^i_2, \ldots, b^i_k) \wedge (p^i(1), p^i(2), \ldots, p^i(k))$$

if $b^i \neq \tilde{b}^i$.

The implication of Proposition 1 is illustrated graphically in Figures 3 and 4. If bidder i's preferences are such that she has a demand curve p^i, then the bid b^i as seen in Figure 3 is weakly dominated.

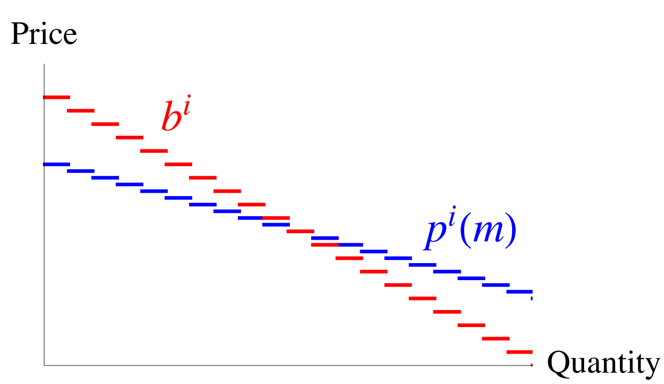

Figure 3: Underreporting in uniform-price auction.

In particular, the bid b^i is weakly dominated by a bid that is represented by the lower envelope of b^i and p^i.

3.2 In the Vickrey auction

Similarly, Proposition 2 shows that in the Vickrey auction, it is a dominated strategy for a bidder to underreport her demand and misreport her demand for her first unit. This result mirrors Proposition 1 and Ausubel et al.'s demand reduction result, but the intuition behind the overbidding is distinct. The argument for why bidders overreport is broken into two cases.

First, consider a case where bidding truthfully is a best reply. In the Vickrey auction, bidder i faces a residual demand curve that is determined by other bidders' reports. This serves as her marginal price curve. Suppose that this residual demand curve was perfectly elastic. That is, the k highest bids of her opponents all exactly equal p. Thus, bidder i pays a constant marginal price p for each unit of the good she acquires. This case is illustrated in Figure 4. When facing a constant marginal price curve, truthful reporting is a best reply. By truthfully reporting demand, bidder i wins her desired number of objects when the price per unit is p. Suppose that she wins x objects in this case.

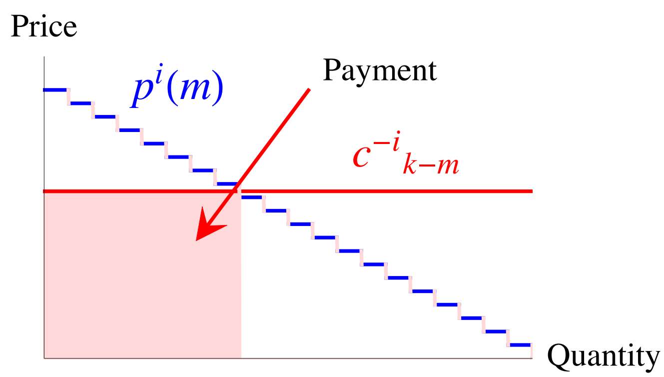

Figure 4: Vickrey payment with perfectly elastic residual demand.

Now, suppose instead that the same bidder faces a relatively less elastic residual demand curve. Thus, she pays an increasing marginal price for each unit she acquires. In addition, suppose that the residual demand is such that bidder i pays p for her x^{th} unit of the good. If bidder i reports her demand truthfully, she will again win x units. However, she does not pay the price p per unit. Instead, p is the price she pays for only x^{th} last unit of the good. She pays a price less than p for all other units. Or equivalently, she pays p per unit and is then given a refund. This is illustrated in Figure 5. Since bidder i has weakly positive wealth effects, the refund weakly increases her demand for acquiring additional units of the good. This effect gives bidder i an incentive to overreport her demand curve. Thus, truthful reporting is not necessarily a best reply to the profile of competing bids.

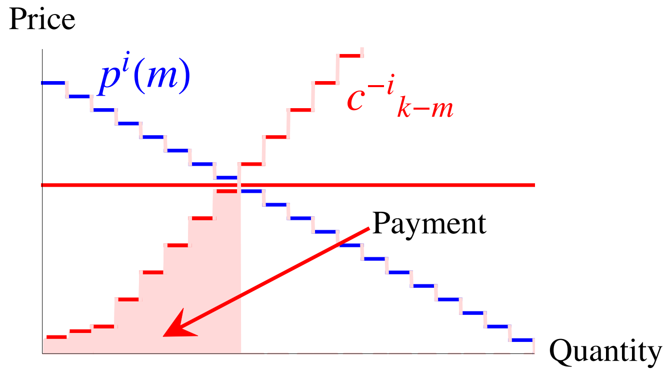

Figure 5: Vickrey payment with more inelastic residual demand

The amount that bidder i should overreport her demand curve depends on the magnitude of the wealth effects and the expected elasticity of the residual demand curve. Underreporting demand however, is weakly dominated.

Proposition 2. (No underreporting in Vickrey auctions)
The bid $b^i = (b_1^i, b_2^i, \ldots, b_k^i) \in \beta$ *is weakly dominated by the bid*

$$\tilde{b}^i = \big((p^i(1), b_2^i, \ldots, b_k^i) \vee (p^i(1), p^i(2), \ldots, p^i(k))\big) \wedge (p^i(1), p^i(1), \ldots, p^i(1))$$

if $b^i \neq \tilde{b}^i$.

In the expression for $\tilde{b}^i$, the final term is needed to ensure that bids are weakly decreasing and $\tilde{b}^i \in \beta$.

Propositions 1 and 2 display similar results. In both auctions bidders truthfully report their demand for their first unit. In the Vickrey auction, bidders never underreport their demands for later units and in the uniform-price auction bidders never overreport their demand for later units. They combine to give Lemma 1.

Lemma 1. (Uniform-price auctions produce more winners)
Suppose that bidders play undominated strategies. For a given auction environment, the uniform-price auction produces a (weakly) greater number of winners than the Vickrey auction. Winners in the Vickrey auction win a (weakly) greater number of objects.

A bidder is more likely not to win any objects in the Vickrey auction. Consider the same set of bidders participating in the uniform-price and Vickrey auctions. There may exist a bidder who does not win any objects in a Vickrey auction, but does in the uniform-price auction. The reverse cannot be true. If a bidder does not win an object in the uniform-price auction, she would not win any objects in the Vickrey auction.

The intuition for Lemma 1 follows from Propositions 1 and 2. Both auctions have the same winning rule and a bidder's first bid is the same in both auctions. However, the distribution of winning bids is higher in the Vickrey auction than in the uniform-price auction because bidders overreport demand in the former and underreport demand in the latter. A bidder will win at least one object if her first bid (this is always her highest bid) ranks among the top k bids. The first bid is more likely to rank in the top k in the uniform-price auction, where the overall distribution of submitted bids is lower. This means that bidders who do win objects in the Vickrey auction win on average a greater number of objects than they would in the uniform-price auction.

If the goods being auctioned are to be used for downstream competition, one may conjecture that uniform-price auctions would make for a more competitive downstream market. However, we would need to explicitly model downstream competition to fully understand its implications for the outcome of the two different auctions.

3.3 An Example: Overbidding and inefficiencies in a Vickrey auction

I use an example to show that overbidding can lead to inefficiencies in the Vickrey auction. In my example, only one bidder has non-quasilinear preferences. While the Vickrey auction is inefficient in this setting, I show that the uniform-price auction implements a Pareto efficient allocation. Thus, the efficiency ranking of the two formats is ambiguous, even if only one bidder has non-quasilinear preferences.

Suppose that there are two bidders and two objects. Bidder 1 has an initial wealth of 100 and preferences represented by

$$u^1(x, w) = 2x + \sqrt{w}.$$

Bidder 2 has quasilinear preferences

$$u^2(1, w) = 30 + w,$$
$$u^2(2, w) = 30 + v + w,$$

where with probability .5, $v = 20$ and with probability .5, $v = 30$. The realization of v is bidder 2's private information. The ex-ante distribution of v is commonly known. I derive bid behavior in this simple setting through iterative elimination of weakly dominated strategies.

Note, bidders' inverse demand curves are

$$p^1(1) = 36 \quad p^1(2) = 29.5,$$
$$p^2(1) = 30 \quad\quad p^2(2) = v.$$

Truthful reporting is a dominant strategy for bidder 2. This is because she has quasilinear preferences and we can apply the standard argument for truthful reporting being a dominant strategy in Vickrey auctions. Thus $b^2 = (30, v)$.

By Proposition 2, bidder 1 reports 36 as her first bid. Since bidder 2 reports truthfully, bidder 1 will pay v to acquire her first unit of the good. If she wins two units of the good, she pays $30 + v$. Thus, if bidder 1 reports the bid $b^1 = (36, b_2^1)$ where $b_2^1 < 30$ she wins one unit and pays v. If she bids $b^1 = (36, b_2^1)$ where $b_2^1 \geq 30$, she wins two objects and pays $30 + v$. It is then a dominant strategy for bidder 1 to report $b^1 = (36, b_2^1)$ where $b_2^1 \geq 30$ because

$$\frac{1}{2}\left(u^1(2, 100 - 50) + u^1(2, 100 - 60)\right) \geq \frac{1}{2}\left(u^1(1, 100 - 20) + u^1(2, 100 - 30)\right).$$

Thus bidder 1 always wins both goods. She pays 50 with probability $\frac{1}{2}$ and 60 with probability $\frac{1}{2}$.

In this example there is a $\frac{1}{2}$ probability that the allocation is inefficient. There is a $\frac{1}{2}$ probability that $v = 30$, and bidder 1 wins both units and pays 30 for each. Thus, bidder 1 receives a payoff of $u^1(2, 40)$. Since bidder 1 demands one object only when the price per unit is 30, then $u^1(2, 100 - 60) < u^1(1, 100 - 30)$. Thus, both bidders are made weakly better off by an allocation which assigns each bidder 1 unit and each bidder pays 30. This allocation gives the same revenue, strictly increases bidder 1's payoff, and leaves bidder 2 equally well-off.

The intuition behind this observation follows from Proposition 2. Bidder 1 is unsure of the shape of the residual demand curve that she faces. She may pay a low price for her first unit (20) or a high price (30). If she expects to pay a lower price for her first unit, she is wealthier and willing to bid more for her second unit. Thus, it is a best response for bidder 1 to overbid on her second unit. However, when bidder 1 pays the higher price for her first unit, she does not want to pay 30 for the second unit, while bidder 2 does. Yet, bidder 1 still wins the second unit for the price of 30.

In this same setting, the uniform-price auction is efficient. I use iterative elimination of weakly dominated strategies to predict bid behavior. Both bidders truthfully report their demand for their first unit. Neither bidder overreports their demand for their second unit, so each bidder wins exactly one unit. The market clearing price is the highest losing bid. Since each bidder wins one object, this is the higher of second bids submitted by either bidder,

$\max\{b_2^1, b_2^2\}$. Thus, it is a weakly dominant strategy for both bidders to set $b_2^1 = b_2^2 = 0$, and the outcome is efficient but gives zero revenue.

The purpose of this example is not to show that the uniform-price auction is in general efficient or yields lower revenues than the Vickrey auction. Ausubel et al. show that even with the quasilinearity restriction, uniform-price auctions are generally inefficient and explicit revenue characterizations are difficult to obtain. Instead, the point of this example is to show that overbidding in Vickrey auctions can lead to inefficiencies and there are cases where the uniform-price auction is efficient and the Vickrey auction is inefficient. Thus, the efficiency ranking of the two auctions is ambiguous, even when only one bidder has non-quasilinear preferences.

4 Conclusion

In this paper, I compare bid behavior in the uniform-price auction and the Vickrey auction. I replace the standard quasilinearity restriction with the more general assumption that the goods being sold are (weakly) normal. Without the quasilinearity restriction, the Vickrey and uniform-price auctions have similar virtues and deficiencies.

Neither auction gives bidders incentives to truthfully report their preferences. Instead, bidders overreport demand in the Vickrey auction and underreport demand in the uniform-price auction. The intuition for the two results is distinct. Ausubel et al.'s (2014) intuition for underbidding in uniform-price auctions holds, even without the quasilinearity restriction. Bidders overreport their demand in the Vickrey auction because of the presence of wealth effects. In the Vickrey auction, a bidder faces an upward sloping supply curve defined by other bidders' reported demands. If a bidders pays a relatively low price for her first unit, positive wealth effects increase her demand for additional units. The bidder's increased demand for additional units gives her an incentive to overreport her demands. The incentive to misreport demands lead to inefficiencies in both auctions, and the efficiency ranking of both auctions is ambiguous.

Moving forward, we could explore how these results relate to pay-as-bid auctions. Another next step would be to look at the efficient auction design problem. If Vickrey auctions are not efficient, then is there an alternative mechanism that is efficient, or possibly a useful second best? In addition, from the practical perspective, it would be interesting to understand the conditions on bidder beliefs and preferences under which the uniform-price auction does better/worse than the Vickrey auction in terms of efficiency.

5 Proofs

Proposition 1

Proof. I simplify notation by letting $U^i(b^i, b^{-i})$ be the utility of bidder i when she bids b^i and her opponents bid b^{-i} in the uniform-price auction. I show that $U^i(\tilde{b}^i, b^{-i}) \geq U^i(b^i, b^{-i})\ \forall b^{-i} \in \beta^{N-1}$. Suppose that the $(N-1)k$ competing bids are described by the vector c^{-i}, where $c_1^{-i} \geq c_2^{-i} \geq \dots c_{(N-1)k}^{-i}$.

Case 1*: bidder i wins the same number of objects if she bids b^i and $\tilde{b}^i$.*

Suppose i wins m objects from bidding either b^i or $\tilde{b}^i$. If $m = 0$ then $U^i(\tilde{b}^i, b^{-i}) = U^i(b^i, b^{-i}) = u^i(0,0)$. If $m \in \{1, \dots k\}$, then $b_m^i, \tilde{b}_m^i \geq c_{k-m+1}^{-i}$ and $c_{k-m}^{-i} \geq b_{m+1}^i, \tilde{b}_{m+1}^i$. If it is the case that $c_{k-m}^{-i} \geq b_{m+1}^i, \tilde{b}_{m+1}^i$, then the market clearing price for either bid is c_{k-m}^{-i}. Thus, $U^i(\tilde{b}^i, b^{-i}) = U^i(b^i, b^{-i}) = u^i(m, w^i - mc_{k-m}^{-i})$. If $b_{m+1}^i, \tilde{b}_{m+1}^i \geq c_{k-m}^{-i}$, then b_{m+1}^i is the market clearing price for either bid. Thus, $U^i(\tilde{b}^i, b^{-i}) = U^i(b^i, b^{-i}) = u^i(m, w^i - mb_{m+1}^i)$. Finally if $b_{m+1}^i \geq c_{k-m}^{-i} > \tilde{b}_{m+1}^i$, then c_{k-m}^{-i} is the market clearing price when bidder i bids $\tilde{b}^i$ and b_{m+1}^i is the market clearing price when bidder i bids b^i. Thus,

$$U^i(\tilde{b}^i, b^{-i}) \geq u^i(m, w^i - mc_{k-m}) > u^i(m, w^i - mb_{m+1}^i) = U^i(b^i, b^{-i}).$$

Case 2*: bidder i wins more objects by bidding $\tilde{b}^i$ instead of b^i.*

This implies that, $p^i(1) = \tilde{b}_1^i > b_1^i$. If not, then $b_j^i \geq \tilde{b}_j^i\ \forall\ j \in \{1, \dots, k\}$ and i wins a (weakly) greater number of objects by bidding b^i instead of $\tilde{b}^i$. Thus, $\tilde{b}^i$ strictly exceeds b^i only in the first dimension, as $p^i(1) = \tilde{b}_1^i > b_1^i \geq b_j^i \geq \tilde{b}_j^i\ \forall j \in \{2, \dots k\}$. If i wins more objects by bidding $\tilde{b}^i$ instead of b^i, she then must win exactly one object by bidding $\tilde{b}^i$ and zero goods by bidding b^i, since she submits a higher bid only for her first good when bidding $\tilde{b}^i$ instead of b^i. Thus,

$$U^i(\tilde{b}^i, b^{-i}) = u^i(1, w^i - \max\{\tilde{b}_2^i, c_k^{-i}\}) \geq u_i(1, w^i - p^i(1)) = u^i(0, w^i) = U^i(b^i, b^{-i}),$$

where the first inequality holds because $p^i(1) = \tilde{b}_1^i \geq c_k^{-i}$ if i wins one object. The equality $u^i(1, w^i - p^i(1)) = u^i(0, w^i)$ holds because of the construction of p^i.

Case 3*: bidder i wins fewer objects by bidding $\tilde{b}^i$ instead of b^i.*

Let m be the number of objects bidder i wins when bidding b^i, and let $\tilde{m}$ be the number bidder i wins when bidding bidding $\tilde{b}^i$. By assumption $m > \tilde{m}$. This implies that $b_n^i > \tilde{b}_n^i = p^i(n)\ \forall n \in \{\tilde{m}+1, \dots, m\}$. If not, then bidder i wins more than $\tilde{m}$ objects by bidding $\tilde{b}^i$. Thus, $c_{k-n}^{-i} \geq b_n^i = p^i(n)$ for all $n \in \{\tilde{m}+1, \dots, m\}$. Since c_{k-n}^{-i} is weakly increasing in n and $p^i(n)$ is weakly decreasing in n, then $c_{k-m+1}^{-i} \geq p^i(\tilde{m}+1)$. If bids $\tilde{b}^i$ and wins $\tilde{m}$ objects, she pays $p_{\tilde{U}} = \max\{\tilde{b}^i_{\tilde{m}+1}, c^{-i}_{k-(\tilde{m}+1)}\}$. Thus $p^i(\tilde{m}) \geq p_{\tilde{U}} \geq p^i(\tilde{m}+1)$. If she bids b^i and wins m objects, she pays p_U where $p_U \geq p_{\tilde{U}}$. By the construction of p^i, it then follows that

$$U^i(\tilde{b}^i, b^{-i}) = u^i(\tilde{m}, w^i - \tilde{m}p_{\tilde{U}}) \geq u^i(m, w^i - mp_{\tilde{U}}) \geq u^i(m, w^i - mp_U) = U^i(b^i, b^{-i}).$$

□

Proposition 2

Proof. Once again, I simplify notation by letting $V^i(b^i, b^{-i})$ be the utility of bidder i when she bids b^i and her opponents bid b^{-i} in the Vickrey auction. I want to show that $V^i(\tilde{b}^i, b^{-i}) \geq V^i(b^i, b^{-i})$ $\forall b^{-i} \in \beta^{N-1}$. Suppose that the $(N-1)k$ competing bids are described by the vector c^{-i}, where $c^{-i}_1 \geq c^{-i}_2 \geq \ldots c^{-i}_{(N-1)k}$.

Case 1: bidder i wins the same number of objects if she bids b^i and $\tilde{b}^i$.

In each case, the bidder pays the same amount. This is because the marginal price of an additional unit is based on other bidders' reports.

Case 2: bidder i wins fewer objects by bidding $\tilde{b}^i$ instead of b^i.

This implies that $p^i(1) = \tilde{b}^i_1 < b^i_1$. If not, then $\tilde{b}^i_j \geq b^i_j$ $\forall j \in \{1, \ldots, k\}$ and i wins (weakly) fewer objects by bidding b^i instead of $\tilde{b}^i$. Thus, b^i strictly exceeds $\tilde{b}^i$ only in the first dimension, as $b^i_1 > p^i(1) = \tilde{b}^i_1 \geq \tilde{b}^i_j \geq b^i_j$ $\forall j \in \{2, \ldots k\}$. If i wins more objects by bidding b^i instead of $\tilde{b}^i$, she then must win exactly one object by bidding b^i and zero goods by bidding $\tilde{b}^i$, since she only submits a higher bid for her first good when bidding $\tilde{b}^i$ instead of b^i. Thus $b^i_1 \geq c^{-i}_k \geq p^i(1)$ and

$$V^i(\tilde{b}^i, b^{-i}) = V^i(0, w^i) = V^i(1, w^i - p^i(1)) \geq V^i(1, w^i - c^{-i}_k) = V^i(b^i, b^{-i}).$$

Case 3: bidder i wins more objects by bidding $\tilde{b}^i$ instead of b^i.

Let m be the number of objects i wins when bidding b^i and $\tilde{m}$ the number she wins when bidding $\tilde{b}^i$. By assumption $\tilde{m} > m$. This implies that $\tilde{b}^i_n = p^i(n) > b^i_n$ $\forall n \in \{m+1, \ldots, \tilde{m}\}$. If not, bidding b^i wins more than m objects. Thus, bidder i pays $\sum_{j=1}^{m} c^{-i}_{k+1-j}$ when she bids b^i and she pays $\sum_{j=1}^{\tilde{m}} c^{-i}_{k+1-j}$ when she bids $\tilde{b}^i$. Notice also that $\tilde{b}^i_{\tilde{m}} = p^i(\tilde{m}) \geq c^{-i}_{k+1-\tilde{m}} \geq c^{-i}_{k+1-j}$ for all $j \in \{1 \ldots \tilde{m}\}$. By the definition of p^i,

$$V^i(\tilde{m}, w^i - \tilde{m}p^i(\tilde{m})) \geq V^i(m, w^i - mp^i(\tilde{m})).$$

Note that $mp^i(\tilde{m}) - \sum_{j=1}^{m} c^{-i}_{k+1-j} \geq 0$ since $p^i(\tilde{m}) \geq c^{-i}_{k+1-j}$ for all $j \in \{1, \ldots, \tilde{m}\}$. Thus, positive wealth effects imply that

$$V^i(\tilde{m}, w^i - \tilde{m}p^i(\tilde{m}) + mp^i(\tilde{m}) - \sum_{j=1}^{m} c_{k+1-j}^{-i}) \geq V^i(m, w^i - mp^i(\tilde{m}) + mp^i(\tilde{m}) - \sum_{j=1}^{m} c_{k+1-j}^{-i}).$$

Which can be rewritten as

$$V^i(\tilde{m}, w^i - \sum_{j=1}^{m} c_{k+1-j}^{-i} - (\tilde{m} - m)p^i(\tilde{m})) \geq V^i(m, w^i - \sum_{j=1}^{m} c_{k+1-j}^{-i}) = V^i(b^i, b_{-i}). \tag{1}$$

Recalling again that $p^i(\tilde{m}) \geq c_{k+1-j}^{-i} \forall j \in \{1, \ldots, \tilde{m}\}$, implies that $(\tilde{m}-m)p^i(\tilde{m}) \geq \sum_{j=m+1}^{\tilde{m}} c_{k+1-j}^{-i}$. Thus, $(\tilde{m} - m)p^i(\tilde{m}) + \sum_{j=1}^{m} c_{k+1-j}^{-i} \geq \sum_{j=1}^{\tilde{m}} c_{k+1-j}^{-i}$. This implies that,

$$V^i(\tilde{b}^i, b_{-i}) = V^i(m, w^i - \sum_{j=1}^{\tilde{m}} c_{k+1-j}^{-i}) \geq V^i\left(\tilde{m}, w^i - \left(\sum_{j=1}^{m} c_{k+1-j}^{-i} + (\tilde{m} - m)p^i(\tilde{m})\right)\right). \tag{2}$$

Combining 1 and 2 implies that $V^i(\tilde{b}^i, b_{-i}) \geq V^i(b^i, b_{-i})$. $\square$

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