Amherst College
Department of Mathematics and Statistics

Comprehensive Examination

◁ Algebra ▷

Practice Exam 1

Number: _____ Solutions

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable. Show all your work, and justify your answers.
- The Algebra Exam consists of Questions 1–4 that total to 100 points.

For Department Use Only:

Grader #1: ____________________

Grader #2: ____________________

1. Let G_1 and G_2 be groups and $\phi : G_1 \to G_2$ be a homomorphism. Suppose that N_2 is a subgroup of G_2 and define the set

$$N_1 = \{a \in G_1 : \phi(a) \in N_2\}.$$

(a) Prove that N_1 is a subgroup of G_1.
[Note: this is a standard theorem in Math 350. Since you are being asked to prove that theorem here, you may not quote that theorem.]

- Let e_1 be the identity in G_1 and e_2 be the identity in G_2. Then $\varphi(e_1) = e_2$ since φ is a homomorphism. Since N_2 is a subgroup of G_2, $e_2 \in N_2$ and so $e_1 \in N_1$.
- Let $a, b \in N_1$ so that $\varphi(a), \varphi(b) \in N_2$. Then $\varphi(ab) = \varphi(a)\varphi(b) \in N_2$ since N_2 is closed. Hence, $ab \in N_1$ and so N_1 is also closed.
- Finally, if $a \in N_1$ then $\varphi(a) \in N_2$ and so $\varphi(a)^{-1} \in N_2$ since N_2 is a subgroup. Since φ is a homomorphism, $\varphi(a^{-1}) = \varphi(a)^{-1} \in N_2$. Thus, $a^{-1} \in N_1$.

Therefore, N_1 is a subgroup of G_1.

(b) Prove that if N_2 is a normal subgroup of G_2 then N_1 is a normal subgroup of G_1.

Suppose that N_2 is a normal subgroup of G_2. By part a, we know that N_1 is a subgroup of G_1. Let $g \in G_1$. We will show that $\forall n \in N_1$, $gng^{-1} \in N_1$. To see this, consider $\varphi(gng^{-1}) = \varphi(g)\varphi(n)\varphi(g^{-1})$

$$= \varphi(g)\varphi(n)\varphi(g)^{-1}.$$

Since $n \in N_1$, $\varphi(n) \in N_2$. Then, since $\varphi(g) \in G_2$ and N_2 is normal in G_2, we have that $\varphi(g)\varphi(n)\varphi(g)^{-1} \in N_2$. Hence, $\varphi(gng^{-1}) \in N_2$ and so $gng^{-1} \in N_1$ by definition. Thus, N_1 is normal in G_1.

2. Let G be a finite group. Suppose that x and y are distinct elements of order two in G such that $xy = yx$. Prove that the order of G is divisible by 4.

Let G be a finite group and $x, y \in G$ such that $x \neq y$, x and y both have order 2, and $xy = yx$. Consider the set $H = \{x, y, xy, e\}$. We claim that H is a subgroup of G and the order of H is 4.

To see that H is a subgroup, first note that $e \in H$ so clearly $H \neq \phi$.

Next we will show that H is closed by computing the 16 possible products of the 4 elements of H. Indeed, we have

1. $x \cdot x = x^2 = e$ since $o(x) = 2$
2. $x \cdot y \in H$ by defintion
3. $x(xy) = x^2 y = ey = y$
4. $x \cdot e = x$
5. $y \cdot x = xy$ by assumption
6. $y \cdot y = y^2 = e$ since $o(y) = 2$
7. $y \cdot xy = y \cdot yx = y^2 x = ex = x$
8. $y \cdot e = y$
9. $xy \cdot x = yx \cdot x = yx^2 = ye = y$
10. $xy \cdot y = xy^2 = xe = x$
11. $xy \cdot e = xy$
12. $(xy)(xy) = xyyx = xy^2x = xex = x^2 = e$
13. $ex = e$
14. $e \cdot y = y$
15. $e \cdot xy = xy$
16. $ee = e$

Finally, we show that H is closed under inverses. Since $x^2 = x \cdot x = e$, $x^{-1} = x \in H$. Similarly, we have $y^2 = e$ and $(xy)^2 = e$ (from 12 above) so $y^{-1} = y \in H$ and $(xy)^{-1} = xy \in H$. Thus, H is a subgroup of G.

It remains to show that H contains exactly four distinct elements. We already know that $x \neq y$ and since x and y have order 2, neither is equal to the identity. Then since $x \neq e$, $xy \neq y$ and similarly $xy \neq x$. Finally, since $x = x^{-1}$, we have $x^{-1} \neq y$ and so $xy \neq e$. Thus, $|H| = 4$.

Since the order of a subgroup must divide the order of the group, the order of G must be divisible by 4.

3. Suppose that σ is a permutation in the **alternating group** A_{10} given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 7 & 2 & 6 & 10 & 1 & 5 & & & 3 \end{pmatrix}$$

where the images of 8 and 9 have been lost.

(a) Determine the images of 8 and 9 under σ. Don't forget to justify your answer.

First write $\sigma = (1,4,6)(2,7,5,10,3)\tau$ where τ is the unknown cycle or product of cycles containing 8 and 9. Note that since all other numbers have already been determined as the image of 1-7, and 10, we may assume that τ is disjoint from the other two cycles in σ. The only possibilities are $\tau = (8,9)$ or $\tau = (8)(9)$. In the first case, τ has length 2 and hence is an odd permutation. In the second case, τ is the identity on 8 and 9 and so is even. Now $(1,4,6)$ has length 3, so it is an even cycle and $(2,7,5,10,3)$ has length 5, so it is also even. Then, the product $(1,4,6)(2,7,5,10,3)$ is even as well. If τ were odd, it would then follow that σ would be odd. However, $\sigma \in A_{10}$ must be even. Thus, $\tau = (8)(9)$ is the only possibility. So, $\sigma(8) = 8$ and $\sigma(9) = 9$.

(b) Compute the order of σ.

We determined above that σ can be written as a product of two disjoint cycles of lengths 3 and 5. So, the order of σ is $\text{lcm}(3,5) = 15$.

4. Let R be a ring.

   (a) Define what it means for a subset $I \subseteq R$ to be an **ideal** of R.

   If you use any other technical terms like "closed," "subring," "group," "subgroup," etc., you must fully define those terms as well.

$I \subseteq R$ is an ideal of R if and only if

1. $I \neq \emptyset$,
2. $\forall x, y \in I$, $x - y \in I$,

and 3. $\forall x \in I$ and $r \in R$, rx and xr are in I.

   (b) Let $R = \mathbb{R}[x]$ be the ring of polynomials with coefficients in the field $\mathbb{R}$ of real numbers. Let $I \subseteq R$ be the subset

$$I = \{f \in \mathbb{R}[x] : f(1) = f(2) = 0\}.$$

   Prove that I is an ideal of R.

1. First note that the zero polynomial $0(x) := 0 \ \forall x \in \mathbb{R}$ satisfies $0(1) = 0(2) = 0$ and so $0(x) \in I$.
2. Next let $f, g \in I$ so that $f, g \in \mathbb{R}[x]$ and $f(1) = f(2) = g(1) = g(2) = 0$. Then $f-g \in \mathbb{R}[x]$ and $(f-g)(1) = f(1) - g(1) = 0$ and $(f-g)(2) = f(2) - g(2) = 0$ so $f - g \in I$.
3. Finally, let $f \in I$ and $r \in \mathbb{R}[x]$. Then fr and $rf \in \mathbb{R}[x]$ since ~~and~~ the product of two polynomials is a polynomial. And $(rf)(1) = r(1)f(1) = r(1) \cdot 0 = 0$, $(fr)(1) = f(1)r(1) = 0 \cdot r(1) = 0$.

   So, rf and fr are in I.

Thus, I is an ideal of R.

   (c) Prove that R/I has zero-divisors. That is, show that there are two nonzero elements of R/I whose product is zero in R/I. Note: $R = \mathbb{R}[x]$ and I is as above!

Consider $f(x) := x - 1$ and $g(x) := x - 2$. Then $f(2) = 1$ so $f \notin I$ and hence $f + I \neq 0 + I$. Similarly, $g(1) = -1$ so $g \notin I$ and $g + I \neq 0 + I$. But $(fg)(x) = (x-1)(x-2)$ satisfies $(fg)(1) = 0 = (fg)(2)$. Hence, $fg \in I$, and so $0 + I = fg + I = (f+I)(g+I)$. Thus, f and g are zero-divisors.