Amherst College
Department of Mathematics and Statistics

COMPREHENSIVE EXAMINATION

◁ ANALYSIS ▷

PRACTICE EXAM 1

NUMBER: _____ Solutions

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable. Show all your work, and justify your answers.
- The Analysis Exam consists of Questions 1–4 that total to 100 points.

For Department Use Only:

GRADER #1: ________________

GRADER #2: ________________

1. (a) State the Axiom of Completeness.

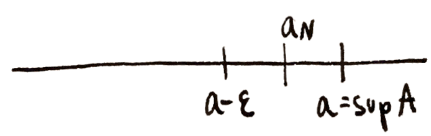
Every nonempty set of real numbers that is bounded above has a least upper bound.

(b) Let (a_n) be a sequence of real numbers. State the ϵ-N definition of what it means for (a_n) to converge to $a \in \mathbf{R}$.

A sequence (a_n) converges to $a \in \mathbb{R}$ if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ such that $\forall n \geq N$, $|a_n - a| < \varepsilon$.

(c) Let (a_n) be an increasing sequence of real numbers and suppose that there exists a real number $M \in \mathbf{R}$ such that $a_n \leq M$ for all n. Use the Axiom of Completeness and the definition in part (b) to prove that the sequence (a_n) converges.

Let (a_n) be a sequence of real numbers with $a_n \leq a_{n+1}\ \forall n$. Suppose that $a_n \leq M\ \forall n$. Since $A = \{a_n\}$ is non-empty and bounded above, by the AOC $\sup A =: a$ exists. We claim that $(a_n) \to a$. To see this, let $\varepsilon > 0$ be arbitrary. Since $a = \sup A$, there exists an element $a_N \in A$ such that $a_N > a - \varepsilon$.



Let $n \geq N$. Then
$$\begin{aligned} |a_n - a| &= a - a_n && \text{since } a_n \leq a = \sup A\ \forall n \\ &\leq a - a_N && \text{since } a_n \geq a_N \text{ for } n \geq N \\ &< \varepsilon. \end{aligned}$$

Thus, $\forall n \geq N$, $|a_n - a| < \varepsilon$. Hence, (a_n) converges.

2. (a) Let $f\colon A \to \mathbf{R}$ be a function. Using the ϵ-δ definition, define what it means for f to be continuous at $c \in \mathbf{A}$.

A function $f: A \to \mathbb{R}$ is continuous at $c \in A$ if $\forall \varepsilon > 0$ $\exists \delta > 0$ such that if $x \in A$ and $|x-c| < \delta$ then $|f(x) - f(c)| < \varepsilon$.

(b) Suppose that the functions $f, g\colon \mathbf{A} \to \mathbf{R}$ are both continuous at $c \in A$. Prove using the above definition that the function $h : A \to \mathbb{R}$ defined by $h(x) = f(x) + g(x)$ is continuous at c.

Suppose that $f, g : A \to \mathbb{R}$ are continuous at c and set $h(x) = f(x) + g(x)$. Let $\varepsilon > 0$ and choose $\delta_f, \delta_g > 0$ such that $\forall x \in A$ if $|x-c| < \delta_f$ then $|f(x) - f(c)| < \varepsilon/2$ and if $|x-c| < \delta_g$ then $|f(x) - f(c)| < \varepsilon/2$. Let $\delta = \min\{\delta_f, \delta_g\}$. Then if $x \in A$ and $|x-c| < \delta$ we have

$$
\begin{aligned}
|h(x) - h(c)| &= |f(x) + g(x) - f(c) - g(c)| \\
&\leq |f(x) - f(c)| + |g(x) - g(c)| \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon .
\end{aligned}
$$

Hence, $\forall x \in A$ with $|x - c| < \delta$, $|h(x) - h(c)| < \varepsilon$. So, h is continuous at c.

3. Suppose that we have a collection of compact sets $K_\lambda \subset \mathbf{R}$ for all λ in some index set Λ.

   (a) Give a condition that is both necessary and sufficient for a set of real numbers to be compact in $\mathbf{R}$.

A set K of real numbers is compact if and only if K is closed and bounded.

   (b) Use the condition in part (a) to prove that the intersection $K = \bigcap_{\lambda\in\Lambda} K_\lambda$ is compact.

- Since K_λ is compact $\forall \lambda \in \Lambda$, K_λ is closed for each λ. Then, since an arbitrary intersection of closed sets is closed, $\bigcap_{\lambda\in\Lambda} K_\lambda$ is closed.
- Let $\lambda_0 \in \Lambda$. (Note that if $\Lambda = \phi$ then $K = \phi$ is trivially compact.) Since K_{λ_0} is compact, K_{λ_0} is bounded. And since $K = \bigcap_{\lambda\in\Lambda} K_\lambda \subseteq K_{\lambda_0}$, K must be bounded as well.

Therefore, K is closed and bounded in $\mathbb{R}$, and hence compact.

   (c) Give an example to show that the union $\bigcup_{\lambda\in\Lambda} K_\lambda$ is not necessarily compact.

Let $\Lambda = \mathbb{N}$ and $K_n = [-n, n]$. Since each K_n is a closed interval, each K_n is closed. Clearly we have $|x| \leq n$ for all $x \in K_n$ and so K_n is bounded and hence compact $\forall n$.

However, $\bigcup_{n\in\mathbb{N}} K_n = \bigcup_{n=1}^{\infty} [-n, n] = \mathbb{R}$ is not bounded.

So, $\bigcup_{\lambda\in\Lambda} K_\lambda$ is not necessarily compact.

4. Consider the sequence of functions (f_n) where $f_n(x) = \frac{1}{1+n^2x^2}$ for $n \geq 1$.

(a) Prove that (f_n) converges pointwise to a function f on $[0, 1]$.

Let $f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } 0 < x \leq 1. \end{cases}$ We claim that (f_n) converges to f pointwise on $[0,1]$.

To see this, let $\varepsilon > 0$ and $x \in [0,1]$.

- If $x = 0$, we have $f_n(0) = 1$ for all $n \geq 1$. Hence, $|f_n(0) - f(0)| = |1-1| = 0 < \varepsilon$.
- If $0 < x \leq 1$, choose $N \in \mathbb{N}$ such that $\frac{\varepsilon^{-1}-1}{x^2} < N^2$. Then for $n \geq N$, we have

$$\frac{\varepsilon^{-1}-1}{x^2} < N^2 \leq n^2$$
$$\Rightarrow \varepsilon^{-1} - 1 < n^2x^2$$
$$\Rightarrow \varepsilon^{-1} < n^2x^2 + 1$$
$$\Rightarrow (1+n^2x^2)^{-1} < \varepsilon.$$

So $|f_n(x) - f(x)| = \left|\frac{1}{1+n^2x^2}\right| = \frac{1}{1+n^2x^2} < \varepsilon$.

Thus, $\forall x \in [0,1]$, $\exists N \in \mathbb{N}$ such that if $n \geq N$, $|f_n(x) - f(x)| < \varepsilon$. So, (f_n) converges to f pointwise.

(b) Prove that (f_n) does not converge uniformly on $[0, 1]$.

Suppose that (f_n) converged uniformly on $[0,1]$. Then since each function $f_n(x) = \frac{1}{1+n^2x^2}$ is continuous on $[0,1]$, the limit function must be continuous on $[0,1]$ as well. But from part a, the limit of (f_n) is $f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } 0 < x \leq 1 \end{cases}$, which is not continuous at $x = 0$.

Thus, (f_n) cannot converge uniformly on $[0,1]$.