Amherst College
Department of Mathematics and Statistics

Comprehensive Examination

◁ Multivariable Calculus and Linear Algebra ▷

Practice Exam 1

Number: _____

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable. Show all your work, and justify your answers.
- The Multivariable Calculus and Linear Algebra Exam consists of Questions 1–8 that total to 200 points.

For Department Use Only:

Grader #1: ________________

Grader #2: ________________

1. Consider the surface S given by $\overbrace{x^2y - yz^2 + z}^{F} = 1$.

   (a) Find an equation of the tangent plane to S at the point $(11, 0, 1)$.

Let $F(x,y,z) = x^2y - yz^2 + z$ so S is given by $F(x,y,z) = 1$.

$\nabla F(x,y,z) = \langle 2xy,\ x^2 - z^2,\ -2yz + 1 \rangle$

$\nabla F(11,0,1) = \langle 0,\ 120,\ 1 \rangle$

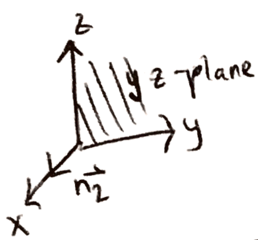
So, the tangent plane is given by

$$0(x-11) + 120(y-0) + 1(z-1) = 0$$

or

$$120y + z = 1$$

   (b) Find two points on the surface S where the tangent plane is parallel to the yz-plane.



If the tangent plane to S is parallel to the yz-plane, the normal vectors $\vec{n_1} = \nabla F(x,y,z)$ and $\vec{n_2} = \langle 1,0,0 \rangle$ are parallel.

Thus, we have $\nabla F(x,y,z) = \lambda \langle 1,0,0 \rangle$ for some $\lambda \in \mathbb{R}$. This gives:

$2xy = \lambda$

$x^2 - z^2 = 0 \Rightarrow x^2 = z^2$

$-2yz + 1 = 0 \Rightarrow 2yz = 1$

We also need $x^2y - yz^2 + z = 1$

Substituting $x^2 = z^2 \Rightarrow z = 1$

Then $2yz = 1 \Rightarrow y = 1/2$

And $x^2 = z^2 \Rightarrow x = \pm z = \pm 1$

So, the two points are $(1, 1/2, 1)$ and $(-1, 1/2, 1)$.

2. Let $f(x, y) = 3x^2 - 3xy + y^3$. Find all critical points of f, and classify each as a local maximum, local minimum, or saddle point.

$f_x(x,y) = 6x - 3y = 0$　　$f_y(x,y) = -3x + 3y^2 = 0$

$\Rightarrow \quad y = 2x$　　$\Rightarrow \quad x = y^2$

Substituting,　$x = (2x)^2 = 4x^2$

$x = 4x^2$

$\Rightarrow x = 0$ or $x = \frac{1}{4}$

$y = 0$　　$y = 1/2$

Critical points: $(0,0)$ and $(1/4, 1/2)$.

$$D(x,y) = \begin{vmatrix} 6 & -3 \\ -3 & 6y \end{vmatrix} = 36y - 9$$

- At $(0,0)$, $D(0,0) = -9 < 0$ so f has a saddle point at $(0,0)$.
- At $(\frac{1}{4}, \frac{1}{2})$, $D(\frac{1}{4}, \frac{1}{2}) = 36(\frac{1}{2}) - 9 > 0$ and $f_{xx}(1/4, 1/2) = 6 > 0$

so f has a local minimum at $(1/4, 1/2)$.

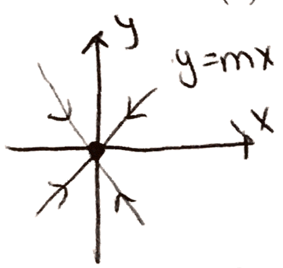
3. Let $f(x,y) = \begin{cases} \dfrac{3x^2y^2}{2x^4+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

(a) Compute $f_x(0,0)$ and $f_y(0,0)$.

$\lim\limits_{h\to 0} \dfrac{f(0+h)-f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{0}{h} = 0$ So $f_x(0,0)$ exists and $f_x(0,0)=0$.

$\lim\limits_{h\to 0} \dfrac{f(0,0+h)-f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{0}{h} = 0$ So $f_y(0,0)$ exists and $f_y(0,0)=0$.

(b) Is f continuous at $(0,0)$? Justify your answer.



Consider the limit of f as (x,y) approaches $(0,0)$ along the straight-line path $y=mx$,

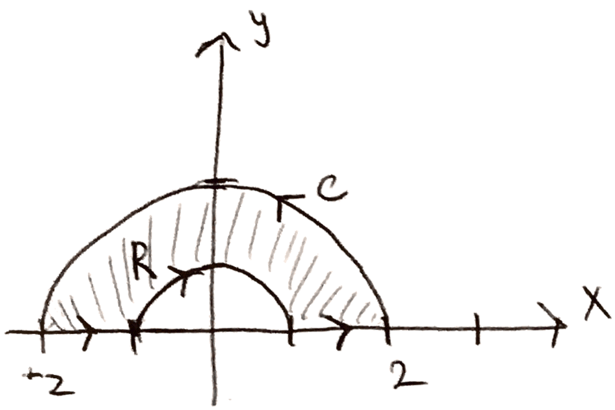
$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} f(x,y) = \lim_{x\to 0} \frac{3x^2(mx)^2}{2x^4+(mx)^4} = \lim_{x\to 0} \frac{3mx^4}{(2+m)x^4} = \frac{3m}{2+m}.$$

So, approaching along the line $y=x$ gives $\lim\limits_{\substack{(x,y)\to(0,0)\\ y=x}} f(x,y) = 1$

while approaching along $y=0$ gives $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$. Since these limits are not equal, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist. Hence, f is not continuous at $(0,0)$.

$P_y = 2y \quad Q_x = 3y$

4. Compute $\int_C y^2\, dx + 3xy\, dy$ where C is the boundary curve of the region bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ in the upper half plane $y \geq 0$, traversed in the counterclockwise direction.
Note. This integral may also be written as $\int_C \langle y^2, 3xy \rangle \cdot d\boldsymbol{r}$



C is a piecewise-smooth, simple, closed, positively oriented curve, so by Green's Theorem, we have:

$$\int_C y^2 dx + 3xy\, dy = \iint_R (3y - 2y)\, dA$$

$$= \int_0^{\pi}\int_1^2 r\sin\theta\, r\, dr\, d\theta$$

$$= \int_0^{\pi} \frac{1}{3} r^3 \Big|_1^2 \sin\theta\, d\theta$$

$$= \int_0^{\pi} \frac{7}{3} \sin\theta\, d\theta$$

$$= -\frac{7}{3}\cos\theta \Big|_0^{\pi}$$

$$= -\frac{7}{3}(-1-1)$$

$$= 14/3.$$

5. (a) Let U and V be subspaces of a vector space W. Prove that $U + V = \{u + v : u \in U, v \in V\}$ is a subspace of W.

- Since U and V are subspaces of W, $0 \in U$ and $0 \in V$. So, $0 + 0 = 0 \in U+V$
- Let $u_1+v_1, u_2+v_2 \in U+V$ for $u_1, u_2 \in U$ and $v_1, v_2 \in V$. Then $(u_1+v_1) + (u_2+v_2) = (u_1+u_2) + (v_1+v_2) \in U+V$ since $u_1+u_2 \in U$ and $v_1+v_2 \in V$.
- Let $u+v \in U+V$ and $c \in \mathbb{R}$. Then $c(u+v) = cu + cv \in U+V$ since $cu \in U$ and $cv \in V$.

So, $U+V$ is a subspace of W.

(b) Suppose $\{u_1, \dots u_m\}$ is a basis for U and $\{v_1, \dots v_n\}$ is a basis for V. Prove that $\{u_1, \dots, u_m, v_1, \dots v_n\}$ spans $U + V$.

Let $w \in U+V$. Then $\exists\, u \in U$ and $v \in V$ such that $w = u+v$.
Since $\{u_1, \dots, u_m\}$ is a basis for U, $\exists\, c_1, \dots, c_m \in \mathbb{R}$ such that $u = c_1u_1 + \dots + c_mu_m$. Similarly, since $\{v_1, \dots, v_n\}$ is a basis for V, $\exists\, d_1, \dots, d_n \in \mathbb{R}$ such that $v = d_1v_1 + \dots + d_nv_n$. Then
$w = u+v = c_1u_1 + \dots + c_mu_m + d_1v_1 + \dots + d_nv_n$, i.e. $\overset{w}{u+v} \in \operatorname{Span}\{u_1, \dots, u_m, v_1, \dots, v_n\}$
Thus, $\{u_1, \dots, u_m, v_1, \dots, v_n\}$ spans $U+V$.

(c) Prove that $\dim(U + V) \leq \dim(U) + \dim(V)$.

By part b, there is a set of $m+n$ vectors that spans $U+V$. Hence, any basis of $U+V$ has at most $m+n$ vectors. Since the dimension of $U+V$ is equal to the number of elements in a basis of $U+V$, we have that $\dim(U+V) \leq m+n$. Since the basis $\{u_1, \dots, u_m\}$ has m elements, $\dim(U) = m$ and since the basis $\{v_1, \dots, v_n\}$ has n elements, $\dim(V) = n$. Therefore, $\dim(U+V) \leq \dim(U) + \dim(V)$.

6. Let V be a vector space.

(a) Explain what it means to say that a subset S of V is a *basis* of V.

A subset S of a vector space V is a basis of V if S is linearly independent and the span of S is V.

(b) Suppose that $\{u, v, w\}$ is a basis of V. Prove that $\{u + 2v, v + 2w, u + 2w\}$ is also a basis of V.

Let $\{u, v, w\}$ be a basis of V and let $c_1, c_2, c_3 \in \mathbb{R}$ be such that $c_1(u+2v) + c_2(v+2w) + c_3(u+2w) = 0$.

Then $(c_1 + c_3)u + (2c_1 + c_2)v + (2c_2 + 2c_3)w = 0$.

Since $\{u, v, w\}$ is linearly independent, we must have

$c_1 + c_2 = 2c_1 + c_2 = 2c_2 + 2c_3 = 0$. Now $c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$ and substituting into $2c_1 + c_2 = 0$ gives $-2c_2 + c_2 = 0 \Rightarrow c_2 = 0$. Hence, $c_1 = -c_2 = 0$ as well. Finally, substituting $c_2 = 0$ into $2c_2 + 2c_3 = 0$ gives $c_3 = 0$. So, $c_1 = c_2 = c_3 = 0$. Therefore, the set $S := \{u+2v, v+2w, u+2w\}$ is linearly independent. Since $\{u, v, w\}$ is a basis for V, $\dim(V) = 3$ and so any set with 3 linearly independent vectors in V must also span V. Thus, S is a basis for V.

7. Consider the matrix $A = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$.

$$\det(A-\lambda I) = \det\begin{pmatrix} 2-\lambda & 1 & -2 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & -1-\lambda \end{pmatrix}.$$

(a) Find all eigenvalues of A.

$$\det(A-\lambda I) = 0 - (1-\lambda)\begin{vmatrix} 2-\lambda & -2 \\ 1 & -1-\lambda \end{vmatrix}$$

$$= -(1-\lambda)\left[(2-\lambda)(-1-\lambda)+2\right]$$

$$= -(1-\lambda)\left[\lambda^2-\lambda-2+2\right]$$

$$= (\lambda-1)(\lambda-1)\lambda$$

Setting $\det(A-\lambda I)=0$ then gives $\lambda = 0, 1$, so the eigenvalues of A are 0 and 1.

(b) Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$, or show that no such matrices exist.

$\underline{\lambda = 0}$ $(A - 0I)v = \vec{0} \Rightarrow \begin{pmatrix} 2 & 1 & -2 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$ $\begin{array}{l} 2x + y - 2z = 0 \\ y = 0 \\ x + y - z = 0 \end{array} \Rightarrow \begin{array}{l} y = 0 \\ x = z \end{array}$ Let $x = z = 1$

So, $v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 0.

$\underline{\lambda = 1}$ $(A - I)v = \vec{0} \Rightarrow \begin{pmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$ $\begin{array}{l} x + y - 2z = 0 \\ 0 = 0 \\ x + y - 2z = 0 \end{array}$ $x = 2z - y$ $\Rightarrow$ • Let $z = 0, y = -1 \Rightarrow x = 1$ • If $z = 1, y = 2 \Rightarrow x = 0$

Then $v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ are both eigenvectors with eigenvalue 1, and they are linearly independent.

Since A has 3 linearly independent eigenvectors, A is diagonalizable and we can write $A = PDP^{-1}$ where $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.

8. Let $P_2 = \{a + bt + ct^2 : a, b, c \in \mathbb{R}\}$ and $T : P_2 \to \mathbb{R}^2$ be defined by

$$T(p) = \begin{bmatrix} p(1) \\ p(2) \end{bmatrix}.$$

(a) Prove that T is linear.

Let $p, q \in P_2$ and $c \in \mathbb{R}$. Then:

- $T(cp) = \begin{bmatrix} c\,p(1) \\ c\,p(2) \end{bmatrix} = c\begin{bmatrix} p(1) \\ p(2) \end{bmatrix} = c\,T(p)$
- $T(p+q) = \begin{bmatrix} (p+q)(1) \\ (p+q)(2) \end{bmatrix} = \begin{bmatrix} p(1)+q(1) \\ p(2)+q(2) \end{bmatrix} = \begin{bmatrix} p(1) \\ p(2) \end{bmatrix} + \begin{bmatrix} q(1) \\ q(2) \end{bmatrix} = T(p) + T(q)$.

Thus, T is linear.

(b) Find the matrix representation of T with respect to the bases $\{1, t, t^2\} =: \beta$ and $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} =: \alpha$

$T(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \begin{matrix} c_1 - c_2 = 1 \\ c_2 = 1 \end{matrix} \Rightarrow c_1 = 2.$

So, $[T(1)]_\alpha = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$T(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \begin{matrix} c_1 - c_2 = 1 \\ c_2 = 2 \end{matrix} \Rightarrow c_1 = 3$

So, $[T(t)]_\alpha = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

$T(t^2) = \begin{bmatrix} 1 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \begin{matrix} c_1 - c_2 = 1 \\ c_2 = 4 \end{matrix} \Rightarrow c_1 = 5$ and $[T(t^2)]_\alpha = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$.

Then the matrix of T with respect to the bases β and α is

$$[T]_\beta^\alpha = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 2 & 4 \end{bmatrix}.$$

(c) Find the rank and nullity of T.

Row reducing $[T]_\beta^\alpha$ we find $\begin{bmatrix} 2 & 3 & 5 \\ 1 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 5 \\ 0 & -1 & -3 \end{bmatrix}$ which has 2 pivots.

Thus, the rank of T is 2. Since the dimension of P_2 is 3, by the rank-nullity theorem, the nullity of T is 1.