Amherst College
Department of Mathematics and Statistics

COMPREHENSIVE EXAMINATION

◁ ALGEBRA ▷

PRACTICE EXAM 2

NUMBER: _____ Solutions

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable. Show all your work, and justify your answers.
- The Algebra Exam consists of Questions 1–4 that total to 100 points.

For Department Use Only:

GRADER #1: ________________

GRADER #2: ________________

1. Let G_1 and G_2 be finite groups and $\phi : G_1 \to G_2$ be a homomorphism. Suppose that $x \in G_1$ has order $n \geq 1$.

   (a) Show that the order of $\phi(x)$ divides n.

Since $x_n = e_1$, $\phi(x)^n = \phi(x^n) = \phi(e_1) = e_2$.

So, if $m = o(\phi(x))$, we have $1 \leq m \leq n$, and we can write $n = km + r$ where $r, k \in \mathbb{N}$ and $0 \leq r \leq m-1$.

Then $e_2 = \phi(x^n) = \phi(x^{km+r}) = \phi(x^{km})\,\phi(x^r) = \left(\left[\phi(x)\right]^m\right)^k \cdot \phi(x^r)$.

Since $m = o(\phi(x))$, $\phi(x)^m = e_2$ and so we have $e_2 = e_2 \cdot \phi(x^r)$, or $\phi(x)^r = e_2$. Since $m = o(\phi(x))$ and $r < m$, we must have $r = 0$.

Thus, $n = km$. In other words, $m \mid n$.

   (b) Prove that if the order of G_2 is relatively prime to n, then x is in the kernel of ϕ.

Suppose that $\gcd(|G_2|, n) = 1$.

Consider $\langle \phi(x) \rangle$. Since $o(\phi(x)) = m$, $|\langle \phi(x) \rangle| = m$ as well.

Since $\langle \phi(x) \rangle$ is a subgroup of the finite group G_2, we have $m \mid |G_2|$.

From part a, we also have $m \mid n$. Hence, $m \mid \gcd(|G_2|, n) = 1$.

Thus, $m = 1$. In other words, $\phi(x)^1 = e_2$, so x is in the kernel of ϕ.

2. Let G be a group and define $Z = \{g \in G : ga = ag \text{ for all } a \in G\}$.

(a) Show that Z is a subgroup of G.

- Since $ea = ae = a$ $\forall a \in G$, $e \in Z$.
- Let $g, h \in Z$. Then $ga = ag$ and $ha = ah$ $\forall a \in G$. So, for any $a \in G$ we have $(gh)a = g(ha) = g(ah) = (ga)h = (ag)h = a(gh)$. then $gh \in Z$.
- Let $g \in Z$. Then for any $a \in G$, $ga = ag$ implies that $a = g^{-1}ag$ and so $ag^{-1} = g^{-1}a$. So, $g^{-1} \in Z$ as well.

Thus, Z is a subgroup of G.

(b) Show that the subgroup Z is normal in G.

Let $g \in G$ and $z \in Z$. Then for any $a \in G$ we have
$$\begin{aligned} gzg^{-1}a &= gg^{-1}za \quad \text{since } z \in Z \\ &= za \\ &= az \quad \text{since } z \in Z \\ &= azgg^{-1} \\ &= agzg^{-1}. \end{aligned}$$

Hence, $gzg^{-1} \in Z$. So, Z is normal in G.

(c) Prove that if the quotient group G/Z is cyclic, then G is abelian.

Suppose that $G/Z = \langle gZ \rangle$ is cyclic with generator gZ for some $g \in G$.
Let $a, b \in G$. Then $aZ = g^kZ$ and $bZ = g^\ell Z$ for some $k, \ell \in \mathbb{N}$.
Then $g^{-k}a$ and $g^{-\ell}b$ are elements of Z and so $\exists z_1, z_2 \in Z$ such that $g^{-k}a = z_1$ and $g^{-\ell}b = z_2$. Then
$$\begin{aligned} ab &= (g^kz_1)(g^\ell z_2) \\ &= g^kg^\ell z_1z_2 \quad \text{since } z_1 \in Z \\ &= g^\ell g^k z_1 z_2 \\ &= g^\ell z_2 g^k z_1 \quad \text{since } z_2 \in Z. \\ &= ba. \end{aligned}$$

Thus, G is abelian.

3. Consider the group S_9 of permutations of the set $\{1, 2, 3, \ldots, 9\}$. Let $\sigma, \tau \in S_9$ be the permutations

$$\sigma = (1,4,3)(9,5,7) \quad \text{and} \quad \tau = (3,9)(1,5,8).$$

(a) Write $\tau\sigma^2$ as a product of **disjoint** cycles.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 2 & 4 & 5 & 3 & 6 & 8 & 1 & 7 \end{pmatrix}$$

$$\tau\sigma^2 = (1,9,7,8)(3,4,5)$$

(b) Compute the **order** of each of σ, τ, and $\tau\sigma^2$.

order$(\sigma) = \text{lcm}(3,3) = 3$

order$(\tau) = \text{lcm}(2,3) = 6$

order$(\tau\sigma^2) = \text{lcm}(4,3) = 12$.

(c) Decide whether each of σ, τ, and $\tau\sigma^2$ is an **even** or **odd** permutation; don't forget to justify.

- σ is the product of two disjoint 3-cycles, which are even, hence σ is even.
- τ is the product of a disjoint 2-cycle and 3-cycle. Since 2-cycles are odd and 3-cycles are even, the product τ is odd.
- $\tau\sigma^2$ is the product of an odd 4-cycle and a disjoint even 3-cycle. So, $\tau\sigma^2$ is odd.

4. Let R be a ring.

   (a) Define what it means for a subset $I \subseteq R$ to be an **ideal** of R.

   If you use any other technical terms like "closed," "subring," "group," "subgroup," etc., you must fully define those terms as well.

$I \subseteq R$ is an ideal of R if
1. $I \neq \emptyset$
2. $\forall x, y \in I$, $x - y \in I$, and
3. $\forall x \in I$ and $r \in R$, rx and xr are in I.

   (b) Let $I \subseteq R$ be an ideal of R, S be another ring, and $\phi : R \to S$ be a ring homomorphism. Prove that if ϕ is surjective then $\phi(I) = \{\phi(x) : x \in I\}$ is an ideal of S.

Suppose that $\phi : R \to S$ is surjective.

- Since ϕ is surjective, $0_S \in \phi(I)$, so $\phi(I) \neq \emptyset$.
- Let $\phi(x)$ and $\phi(y)$ be in $\phi(I)$ for $x, y \in I$.

  Then $\phi(x) - \phi(y) = \phi(x-y) \in \phi(I)$ since $x - y \in I$.
- Let $\phi(x) \in \phi(I)$ for $x \in I$ and $s \in S$. Since ϕ is surjective, $\exists r \in R$ such that $\phi(r) = s$. Then

  $$s\phi(x) = \phi(r)\phi(x) = \phi(rx) \in \phi(I) \text{ since } rx \in I.$$

  Similarly, $\phi(x)s = \phi(x)\phi(r) = \phi(xr) \in \phi(I)$.

Thus, $\phi(I)$ is an ideal of S.