Amherst College
Department of Mathematics and Statistics

Comprehensive Examination

◁ Analysis ▷

Practice Exam 2

Number: ______ Solutions

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable. Show all your work, and justify your answers.
- The Analysis Exam consists of Questions 1–4 that total to 100 points.

For Department Use Only:

Grader #1: ______________

Grader #2: ______________

1. Let $S_N = \sum_{k=1}^{N} \frac{1}{k}$.

(a) Use induction to prove that $S_{2^N} \geq \frac{1}{2}(N+2)$ for every $N \geq 0$.

For $N=0$ we have $S_{2^0} = \sum_{k=1}^{1} \frac{1}{k} = 1 \geq \frac{1}{2}(0+2) = 1$.

Suppose that $N \geq 0$ and $S_{2^N} \geq \frac{1}{2}(N+2)$. Then

$$S_{2^{N+1}} = \sum_{k=1}^{2^{N+1}} \frac{1}{k} = \sum_{k=1}^{2^N} \frac{1}{k} + \sum_{k=2^N+1}^{2^N+2^N} \frac{1}{k} = \sum_{k=1}^{2^N} \frac{1}{k} + \sum_{k=1}^{2^N} \frac{1}{2^N+k}$$

$$\geq \frac{1}{2}(N+2) + \sum_{k=1}^{2^N} \frac{1}{2^N+2^N} = \frac{1}{2}(N+2) + \frac{2^N}{2 \cdot 2^N}$$

$$\geq \frac{1}{2}(N+2) + \frac{1}{2}$$

$$\geq \frac{1}{2}((N+1)+2).$$ Hence, $S_{2^{N+1}} \geq \frac{1}{2}((N+1)+2)$. By induction, $S_{2^N} \geq \frac{1}{2}(N+2)$ $\forall N \geq 0$

(b) Use part (a) to prove that $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge.

Suppose that $\sum_{n=1}^{\infty} \frac{1}{n}$ converged. Then the sequence (S_N) of partial sums defined above must converge: call this limit S. Since (S_{2^N}) is a subsequence of the convergent sequence (S_N), (S_{2^N}) must converge to S as well. However, from part a, the subsequence (S_{2^N}) is unbounded and hence does not converge. Thus, $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge either.

2. (a) Let (a_n) be a sequence of real numbers. State the ϵ-N definition of what it means for (a_n) to converge to $a \in \mathbf{R}$.

The sequence (a_n) of real numbers converges to $a \in \mathbb{R}$ if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ whenever $n \geq N$.

(b) State the Bolzano-Weierstrass Theorem.

Suppose that (a_n) is a bounded sequence of real numbers. Then there is a subsequence (a_{n_k}) of (a_n) that converges to some number $a \in \mathbb{R}$.

(c) Let (a_n) be a Cauchy sequence in $\mathbf{R}$. Use the definition in part (a) and the Bolzano-Weierstrass Theorem from part (b) to prove that (a_n) converges.

Let (a_n) be a Cauchy sequence in $\mathbb{R}$. ~~and let $\varepsilon > 0$.~~ Since every Cauchy sequence is bounded, (a_n) is bounded. Thus, by the B-W Theorem, there is a subsequence (a_{n_k}) that converges to some $a \in \mathbb{R}$. We will prove that (a_n) converges to a as well. Let $\varepsilon > 0$. Since (a_n) is Cauchy, $\exists N_1 \in \mathbb{N}$ such that $|a_n - a_m| < \varepsilon/2$ whenever $n, m \geq N_1$. Since $(a_{n_k}) \to a$, $\exists N_2 \in \mathbb{N}$ such that $|a_{n_k} - a| < \varepsilon/2$ whenever $n_k \geq N_2$. Let $N = \max\{N_1, N_2\}$ and choose an element a_{n_j} of the subsequence (a_{n_k}) such that $n_j \geq N$. Then if $n \geq N$ we have

$$|a_n - a| \leq |a_n - a_{n_j}| + |a_{n_j} - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence, $\forall n \geq N$, $|a_n - a| < \varepsilon$. Thus, (a_n) converges to a.

3. (a) State the Intermediate Value Theorem.

Let f be a continuous function on a closed interval $[a,b]$.

Suppose that M is any number between $f(a)$ and $f(b)$.

Then there is a value $c \in (a,b)$ such that $f(c) = M$.

(b) Suppose $f : [-1,1] \to \mathbf{R}$ is continuous and satisfies $f(-1) = f(1)$. Use the Intermediate Value Theorem to prove that there exists a number $\gamma \in [0,1]$ such that $f(\gamma) = f(\gamma - 1)$.

Let $f : [-1,1] \to \mathbb{R}$ be continuous and $f(-1) = f(1)$.

Set $g(x) = f(x) - f(x-1)$ for $0 \le x \le 1$.

Note that since f is continuous on $[-1,1]$, $f(x-1)$ is continuous on $[0,2]$. Hence, the difference $g(x)$ is continuous on $[-1,1] \cap [0,2] = [0,1]$. We also have

$g(0) = f(0) - f(-1) = f(0) - f(1)$ and

$g(1) = f(1) - f(0) = -g(0)$.

Hence, $g(0) = -g(1)$, and there are two possibilites.

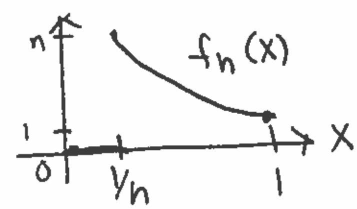
First, if $g(0) = g(1) = 0$ we have $f(0) = f(1)$ and so $f(\gamma) = f(\gamma - 1)$ holds for $\gamma = 1$. Otherwise, $g(0) \neq 0$ implies that $g(0)$ and $g(1)$ have opposite signs. Hence, the value $M = 0$ is between $g(0)$ and $g(1)$. Then by the IVT, $\exists \gamma \in (0,1)$ s.t. $g(\gamma) = 0$.

In this case, we have $g(\gamma) = f(\gamma) - f(\gamma - 1) = 0 \Rightarrow f(\gamma) = f(\gamma - 1)$ as desired.

4. (a) Let (f_n) be a sequence of bounded functions on $[a, b]$. Prove that if (f_n) converges uniformly to f on $[a, b]$, then f is bounded.

Let (f_n) be a sequence of bounded functions on $[a,b]$ and suppose that (f_n) converges uniformly to f on $[a,b]$. Let $\varepsilon = 1$. ~~and~~ Since (f_n) converges unif to f, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ and $\forall x \in [a,b]$ we have $|f_n(x) - f(x)| < 1$. By the reverse triangle inequality (and setting $N = n$) we have $1 > |f_N(x) - f(x)| \geq \big||f_N(x)| - |f(x)|\big| \geq |f(x)| - |f_N(x)|$. Thus, $|f(x)| \leq |f_N(x)| + 1$. Since ~~$f_N(x)$~~ $f_N(x)$ is bounded, $\exists M > 0$ s.t. $|f_N(x)| \leq M$ $\forall x \in [a,b]$. So, $|f(x)| \leq M+1$ $\forall x \in [a,b]$, i.e. f is bounded.

(b) Give an example to show that the statement in part (a) is false if uniform convergence is replaced by pointwise convergence. (Don't forget to prove that each f_n is bounded and that (f_n) converges to f pointwise.)



Let $f_n(x) = \begin{cases} \frac{1}{x} & \frac{1}{n} \leq x \leq 1 \\ 0 & \text{if } 0 \leq x < \frac{1}{n} \end{cases}$

Then $|f_n(x)| \leq n$ $\forall x \in [0,1]$ so each f_n is bounded on $[0,1]$.

And (f_n) converges pointwise to $f(x) = \begin{cases} 1/x & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$ on $[0,1]$

but f is clearly unbounded.