Amherst College
Department of Mathematics and Statistics

Comprehensive Examination

◁ Multivariable Calculus and Linear Algebra ▷

Practice Exam 2

Number: _____ Solutions

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable. Show all your work, and justify your answers.
- The Multivariable Calculus and Linear Algebra Exam consists of Questions 1–8 that total to 200 points.

For Department Use Only:

Grader #1: ________________

Grader #2: ________________

1. Let $F(x, y, z) = x^2 + xy^2 + z$.

(a) Find an equation of the tangent plane to the surface $F(x, y, z) = 4$ at the point $(1, 2, -1)$.

$\nabla F(x, y, z) = \langle 2x + y^2,\ 2xy,\ 1 \rangle$

$\nabla F(1, 2, -1) = \langle 2+4,\ 2\cdot 2,\ 1 \rangle = \langle 6, 4, 1 \rangle = \vec{n}$

T.P. is $6(x-1) + 4(y-2) + (z+1) = 0$

or $6x + 4y + z = 13$

(b) Find the directional derivative of F at the point $(1, 2, -1)$. in the direction of the tangent vector to the curve $\vec{r}(t) = \langle 2t^2 - t, t, t^2 - 2t^3 \rangle$ at $t = 1$.

T.V. $\vec{r}\,'(t) = \langle 4t - 1, 1, 2t - 6t^2 \rangle$

$\vec{r}\,'(1) = \langle 3, 1, -4 \rangle$

$\|\vec{r}\,'(1)\| = \sqrt{9 + 1 + 16} = \sqrt{26}$

$\vec{u} = \frac{1}{\sqrt{26}} \langle 3, 1, -4 \rangle$

$\nabla F(1, 2, -1) = \langle 6, 4, 1 \rangle$ from part a.

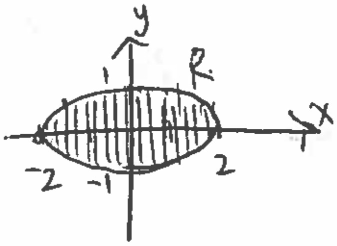
So, $D_{\vec{u}} F(1, 2, -1) = \nabla F(1, 2, -1) \cdot \vec{u}$

$= \langle 6, 4, 1 \rangle \cdot \frac{1}{\sqrt{26}} \langle 3, 1, -4 \rangle$

$= \frac{1}{\sqrt{26}} (18 + 4 - 4)$

$= \frac{18}{\sqrt{26}} = \frac{18\sqrt{26}}{26} = \frac{9\sqrt{26}}{13}.$

2. Find the points at which the absolute maximum and minimum values of the function $f(x,y) = x^2 + 2y^2 + 5$ on the region $x^2 + 4y^2 \le 4$ occur. State all points where the extrema occur as well as the maximum and minimum values.



Interior $f_x(x,y) = 2x = 0 \Rightarrow x = 0$

$f_y(x,y) = 4y = 0 \Rightarrow y = 0$

critical point $(0,0)$ and $f(0,0) = 5$.

Boundary $x^2+4y^2=4$ → let $g(x,y) = x^2+4y^2$ so that

$\nabla g(x,y) = \langle 2x, 8y \rangle$

Then $\nabla f = \lambda \nabla g \Rightarrow 2x = \lambda 2x$ and $4y = \lambda 8y$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$x=0$ or $\lambda = 1$ $\qquad$ $y=0$ or $\lambda = \frac{1}{2}$

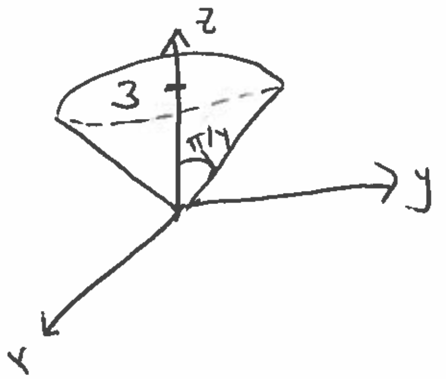
$4y^2=4$ $\quad$ $4y=8y$ $\qquad$ $x^2=4$ $\quad$ $2x=x$

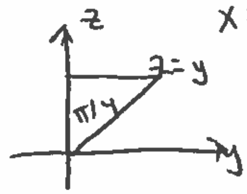
$y=\pm 1$ $\quad$ $y=0$ $\qquad$ $x=\pm 2$ $\quad$ $x=0$

$f(0,\pm 1) = 7$ and $f(\pm 2, 0) = 9$

So, the absolute maximum value on R is 9, which occurs at the points $(2,0)$ and $(-2,0)$. The absolute minimum value is 5, which occurs at $(0,0)$.

3. Calculate the volume of the region that lies both inside the sphere $x^2 + y^2 + z^2 = 9$ and above the cone $z = \sqrt{x^2 + y^2}$.





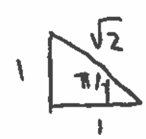
$x = 0 \Rightarrow z = \sqrt{y^2}$

$$V = \iiint_R 1 \, dV$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \int_0^3 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \frac{1}{3}\rho^3 \sin\phi \Big|_{\rho=0}^{\rho=3} d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/4} 9 \sin\phi \, d\phi \, d\theta$$



$$= \int_0^{2\pi} -9\cos\phi \Big|_{\phi=0}^{\phi=\pi/4} d\theta$$

$$= \int_0^{2\pi} \left(-9\left(\frac{1}{\sqrt{2}}\right) + 9\right) d\theta$$

$$= 9 \cdot 2\pi \left(1 - \frac{1}{\sqrt{2}}\right)$$

$$= 18\pi\left(\frac{2-\sqrt{2}}{2}\right) = 9\pi(2-\sqrt{2})$$

$P_y = 3x^2$ $\quad Q_x = 3x^2$ $\quad$ simple, closed, smooth

4. Compute $\int_C 3x^2y\,dx + (x^3 + e^y)\,dy$ where C is the circle $x^2 + y^2 + x = 1$, traversed in the counterclockwise direction.
Note. This integral may also be written as $\int_C \langle 3x^2y, x^3 + e^y\rangle \cdot dr$

$$\int_C 3x^2y\,dx + (x^3+e^y)\,dy = \iint_D (3x^2 - 3x^2)\,dA \quad \text{by Green's theorem}$$

$$= 0$$

(Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, we could also use the Fundamental Thm of Line Integrals.)

5. Let $M_n(\mathbb{R})$ be the vector space of all $n \times n$ matrices with real coefficients. We say that $A, B \in M_n(\mathbb{R})$ commute if $AB = BA$.

(a) Fix $A \in M_n(\mathbb{R})$. Prove that the set of all matrices in $M_n(\mathbb{R})$ that commute with A is a subspace of $M_n(\mathbb{R})$.

Call this set of matrices that commute with A, $C(A)$. Let $0 = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \in M_n(\mathbb{R})$

Since $0A = A0 = 0$, $0 \in C(A)$, so $C(A) \neq \emptyset$.

Let $B, C \in C(A)$ and $d \in \mathbb{R}$. Then

$$\begin{aligned} A(dB+C) &= A(dB) + AC \\ &= dAB + CA \quad \text{since } C \in C(A) \\ &= dBA + CA \quad \text{since } B \in C(A) \\ &= (dB+C)A \end{aligned}$$

So, $dB + C \in C(A)$. Thus, $C(A)$ is a subspace of $M_n(\mathbb{R})$.

(b) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in M_2(\mathbb{R})$ and let $W \subseteq M_2(\mathbb{R})$ be the subspace of all matrices in $M_2(\mathbb{R})$ that commute with A. Find a basis of W.

Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in W$. Then $AB = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ a+c & b+d \end{pmatrix}$

and $BA = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a+b & a+b \\ c+d & c+d \end{pmatrix}$. Since $AB = BA$, we must have

$a+b = a+c \Rightarrow b = c$, $a+b = b+d \Rightarrow a = d$, $a+c = c+d \Rightarrow a = d$ and $c+d = b+d \Rightarrow c = b$. So, $B \in W$ if $b = c$ and $a = d$, i.e. $B = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$.

$B = \begin{pmatrix} a & b \\ b & a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. So, $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$ spans W.

Furthermore, if $a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, we must have $a = b = 0$. So,

$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$ is a linearly independent set that spans W, and hence is a basis of W.

6. Let V and W be vector spaces and let T be a linear transformation from V to W.

(a) Prove that the kernel of T (also called the null space of T) is a subspace of V.

- Since $T(0_V) = 0_W$ since T is linear, $0_V \in \ker T$. Then $\ker T \neq \emptyset$.
- Let $x, y \in \ker(T)$ and $c \in \mathbb{R}$. Then

$$T(cx+y) = cT(x) + T(y) \text{ by linearity}$$
$$= c\,0_W + 0_W$$
$$= 0_W.$$

So, $cx+y \in \ker(T)$. Thus, $\ker(T)$ is a subspace of V.

(b) Prove that T is one-to-one if and only if the kernel of T is $\{0\}$.

Suppose that $\ker(T) = \{0\}$. Then if $u, v \in V$ and $Tu = Tv$ we have $Tu - Tv = 0 \Rightarrow T(u-v) = 0$. So, $u - v \in \ker(T)$. Hence, $u - v = 0$, i.e. $u = v$. So, T is one-to-one.

Now Suppose that T is one-to-one and let $v \in \ker(T)$. Then $T(v) = 0$ but $T(0) = 0$ since T is linear. Thus, $v = 0$, so $\ker(T) = \{0\}$.

(c) Suppose that T is one-to-one and $\{v_1, v_2, \ldots v_n\}$ is a set of n linearly independent vectors in V. Prove that $\{T(v_1), T(v_2), \ldots T(v_n)\}$ is linearly independent in W.

Suppose that $c_1 T(v_1) + \ldots + c_n T(v_n) = 0$. Then by linearity, we have $T(c_1 v_1 + \ldots + c_n v_n) = 0$. So, $c_1 v_1 + \ldots + c_n v_n \in \ker(T)$. Since T is one-to-one, $\ker(T) = \{0\}$ and so $c_1 v_1 + \ldots + c_n v_n = 0$. Since $\{v_1, \ldots, v_n\}$ is linearly independent, $c_1 = c_2 = \ldots = c_n = 0$. Thus, $\{T(v_1), \ldots, T(v_n)\}$ is linearly independent as well.

7. Let P_n be the vector space of polynomials in x with real coefficients of degree at most n. Define $T : P_2 \to P_3$ by $T(f) = \int_0^x f(t)\,dt$. You may assume that T is linear.

(a) Compute $T(a + bx + cx^2)$ where $a, b, c \in \mathbb{R}$.

$$T(a+bx+cx^2) = \int_0^x (a+bt+ct^2)\,dt$$

$$= ax + \frac{1}{2}bx^2 + \frac{1}{3}cx^3 .$$

(b) Compute the matrix of T with respect to the bases $\{1, x, x^2\}$ of P_2 and $\{1, x, x^2, x^3\}$ of P_3.
$\alpha=$ $\beta=$

$T(1) = x = 1(x)$

$T(x) = \frac{1}{2}x^2 = \frac{1}{2}(x^2)$

$T(x^2) = \frac{1}{3}(x^3)$

So, $[T]_\alpha^\beta = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$

8. (a) Let V be a vector space and $T : V \to V$ and $U : V \to V$ be linear transformations that commute, i.e. $T \circ U = U \circ T$. Let $v \in V$ be an eigenvector of T such that $U(v) \neq 0$. Prove that $U(v)$ is also an eigenvector of T.

Let $TU = UT$ and $v \in V$ be such that $Tv = \lambda v$, $v \neq 0$, $Uv \neq 0$.

Then $T(Uv) = (TU)v = (UT)v = U(Tv) = U(\lambda v) = \lambda(Uv)$.

Hence, since $Uv \neq 0$, Uv is an eigenvector for T with eigenvalue λ.

(b) Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation that has 0 as an eigenvalue. What are the possible values of the rank of T? Justify your answer.

Suppose that 0 is an eigenvalue of $T : \mathbb{R}^2 \to \mathbb{R}^2$.

Then there is a nonzero vector $v \in \mathbb{R}^2$ such that $Tv = 0v = 0$.

So, $v \in \ker(T)$ and so the nullity of T is at least 1.

Since the nullity of T is at most $\dim(\mathbb{R}^2) = 2$, the possible values of the nullity ~~of~~ T are 1 and 2. Now, the sum of the rank and nullity of T must be the $\dim(\mathbb{R}^2) = 2$.

Hence, the possible values of the rank of T are 0 and 1.