**Amherst College**
**Department of Mathematics and Statistics**

# Comprehensive Examination

## ◁ Algebra ▷

## Practice Exam 3

NUMBER: _____ Solutions

**Read This First:**

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable. Show all your work, and justify your answers.
- The Algebra Exam consists of Questions 1–4 that total to 100 points.

**For Department Use Only:**

GRADER #1: ________________

GRADER #2: ________________

1. Let $G$ be a group and let $I(G) = \{x \in G : x = x^{-1}\}$.

(a) Show that if $G$ is abelian, then $I(G)$ is a subgroup of $G$.

Suppose that $G$ is abelian.

- Since $e = e^{-1}$, $e \in I(G)$.
- Let $x, y \in I(G)$. Then $x = x^{-1}$ and $y = y^{-1}$. So,

$(xy)^{-1} = y^{-1}x^{-1} = yx = xy$ since $G$ is abelian. Then $xy \in I(G)$.

- Let $x \in I(G)$. Then $x = x^{-1}$ and $(x^{-1})^{-1} = x = x^{-1}$, so $x^{-1} \in I(G)$.

Thus, $I(G)$ is a subgroup of $G$.

(b) Show that if $G$ is finite and $I(G) \neq \{e\}$, then $G$ must have even order.

Suppose $G$ is finite and $I(G) \neq \{e\}$. Then, since $e \in I(G)$, there must exist an $x \neq e$, with $x \in I(G)$. Since $x = x^{-1}$, we have $x^2 = e$ and so $x$ has order 2. Since the order of an element must divide the order of a finite group, $G$ must have even order.

(c) Give an example of a group $G$ for which $I(G)$ is not a subgroup of $G$.

$D_3 = \{e, f, f^2, g, fg, f^2g\}$

- $o(f) = 3$
- $o(f^k g) = 2$

so $I(D_3) = \{e, g, fg, f^2g\}$

$fg \cdot g = f \notin I(D_3)$.

2. Let $G$ be a group.

(a) Let $g, h \in G$. Prove that for all integers $k \geq 1$,

$$(h^{-1}gh)^k = h^{-1}g^kh.$$

Clearly $(h^{-1}gh)^1 = h^{-1}g^1h$ so $(h^{-1}gh)^k = h^{-1}g^kh$ for $k=1$.

Suppose that $(h^{-1}gh)^k = h^{-1}g^kh$ for $k \geq 1$. Then

$$\begin{aligned}(h^{-1}gh)^{k+1} &= (h^{-1}gh)^k(h^{-1}gh) \\ &= (h^{-1}g^kh)(h^{-1}gh) \\ &= h^{-1}g^k(hh^{-1})gh \\ &= h^{-1}g^kgh \\ &= h^{-1}g^{k+1}h.\end{aligned}$$

So, by induction $(h^{-1}gh)^k = h^{-1}g^kh \ \forall k \geq 1$.

(b) Let $N$ be a normal subgroup of $G$ and suppose that $N$ is cyclic. Let $H$ be a subgroup of $N$. Prove that $H$ is a normal subgroup of $G$.

Let $N \trianglelefteq G$ be a normal subgroup of $G$ and suppose that $N = \langle n \rangle$ is cyclic with generator $n \in N$. Let $H \leq N$ be a subgroup of $N$ and let $h \in H$ and $g \in G$. We will show that $g^{-1}hg \in H$ to conclude that $N$ is normal in $G$. Since $H \leq N$ is a subgroup of the cyclic group $N = \langle n \rangle$, $H$ is also cyclic and there is an element of $N = \langle n \rangle$ that generates $H$. So, $\exists k \in \mathbb{Z}$ such that $H = \langle n^k \rangle$.

Then, since $h \in H$, $\exists j \in \mathbb{Z}$ such that $h = (n^k)^j$. Hence,

$$g^{-1}hg = g^{-1}n^{kj}g = (g^{-1}ng)^{kj} \text{ by part a.}$$ * see next page

Now $h \in H \leq N$ and $N$ is normal in $G$, so $g^{-1}ng \in N = \langle n \rangle$.

Thus, $\exists \ell \in \mathbb{Z}$ such that $g^{-1}ng = n^\ell$. So,

$$g^{-1}hg = (g^{-1}ng)^{kj} = (n^\ell)^{kj} = (n^k)^{\ell j} \in \langle n^k \rangle = H.$$

Therefore, $H$ is a normal subgroup of $G$.

*For part b, we need to extend the result in part a to $\mathbb{Z}$.

- For $k=0$, we have $(h^{-1}gh)^0 = e = h^{-1}g^0h$
- For $k<0$, we have $(h^{-1}gh)^k = \left[\left[(h^{-1}gh)^k\right]^{-1}\right]^{-1}$

$$= \left[(h^{-1}gh)^{-k}\right]^{-1}$$

$$= (h^{-1}g^{-k}h)^{-1} \quad \text{by a since } -k \geq 1$$

$$= h^{-1}(g^{-k})^{-1}h$$

$$= h^{-1}g^kh$$

Hence, $(h^{-1}gh)^k = h^{-1}g^kh \quad \forall k \in \mathbb{Z}$.

3. Consider the group $S_{12}$ of permutations of the set $\{1, 2, 3, \ldots, 12\}$.

(a) Give an example of an even permutation $\sigma \in S_{12}$. Don't forget to justify your answer.

Let $\sigma = (1,2,3)$. Since $\sigma$ is a cycle of length 3, which is odd, $\sigma$ is even.

(b) Give an example of a permutation $\tau \in S_{12}$ that has order 14. Don't forget to justify your answer.

Let $\tau = (1,2)(3,4,5,6,7,8,9)$.

Then $\tau$ is the product of disjoint cycles of lengths 2 and 7. Thus, the order of $\tau$ is $\operatorname{lcm}(2,7) = 14$.

(c) Prove that no element of $S_{12}$ has order 13.

Suppose that $\sigma \in S_{12}$ has order 13. Then, since $|S_{12}| = 12!$, $S_{12}$ is a finite group and we must have $13 \mid |S_{12}|$, i.e. $13 \mid 12!$ But 13 is prime and $12 < 13$, so $12!$ is not divisible by 13. So, there is no such $\sigma$.

4. Let $R$ be a ring.

(a) Define what it means for a subset $I \subseteq R$ to be an **ideal** of $R$.

If you use any other technical terms like "closed," "subring," "group," "subgroup," etc., you must fully define those terms as well.

A subset $I \subseteq R$ of a ring $R$ is an ideal if

1. $I \neq \emptyset$.
2. $\forall x, y \in I$, $x - y \in I$, and
3. $\forall x \in I$ and $r \in R$, $rx$ and $xr$ are in $I$.

(b) Let $I \subseteq R$ be an ideal of $R$, and suppose that $xy - yx \in I$ for every $x, y \in R$. Prove that the quotient ring $R/I$ is commutative.

Let $x+I, y+I \in R/I$ for $x, y \in R$.

Then $(x+I)(y+I) = xy + I$ and

$(y+I)(x+I) = yx + I$.

Since $xy - yx \in I$ we have $(xy - yx) + I = 0 + I$.

Thus, $xy + I = yx + I$, and so $(x+I)(y+I) = (y+I)(x+I)$

from above. So, $R/I$ is commutative.