Amherst College
Department of Mathematics and Statistics

Comprehensive Examination

◁ Analysis ▷

Practice Exam 3

NUMBER: ______

Solutions

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable. Show all your work, and justify your answers.
- The Analysis Exam consists of Questions 1–4 that total to 100 points.

For Department Use Only:

GRADER #1: ________________

GRADER #2: ________________

1. Consider the sequence (a_n) defined recursively as follows.

$$a_1 = 2 \qquad \text{and} \qquad a_{n+1} = 5 - \frac{4}{a_n} \qquad \text{for} \qquad n \geq 1.$$

(a) Prove that for $n \geq 1$, $a_{n+1} \geq a_n$.

Let $n=1$. Then $a_1 = 2$ and $a_{n+1} = 5 - \frac{4}{2} = 3$ so $a_{n+1} \geq a_1$.

Suppose that $a_{n+1} \geq a_n$ for $n \geq 1$. Then

$$a_{n+2} = 5 - \frac{4}{a_{n+1}} \geq 5 - \frac{4}{a_n} = a_{n+1}.$$

Thus, $a_{n+1} \geq a_n \quad \forall n \geq 1$.

(b) Prove that the sequence (a_n) is bounded from above.

We claim that (a_n) is bounded above by 4. For $n=1$, $a_1 = 2 \leq 4$.

Suppose that $a_n \leq 4$ for $n \geq 1$. Then $a_{n+1} = 5 - \frac{4}{a_n} \leq 5 - \frac{4}{4} = 4$

So, $a_n \leq 4 \quad \forall n \geq 1$.

(c) Prove that the sequence (a_n) converges and find $\lim_{n\to\infty} a_n$.

Since (a_n) is increasing and bounded from above, (a_n) converges by the Monotone Convergence Theorem. Let $L = \lim_{n\to\infty} (a_n)$.

Note that since (a_n) is increasing and $a_1 = 2$, $L \neq 0$.

then by the Algebraic limit threorem, $\lim_{n\to\infty} \left(5 - \frac{4}{a_n}\right) = 5 - \frac{4}{L}$. Since (a_{n+1}) is a subsequence of (a_n), $\lim_{n\to\infty} a_{n+1} = L$ as well. Thus, we have $5 - \frac{4}{L} = L$

and so $L^2 - 5L + 4 = 0 \Rightarrow (L-4)(L-1) = 0 \Rightarrow L = 1$ or 4.

Again since (a_n) is increasing and $a_1 = 2$, $L \neq 1$. So, $\lim_{n\to\infty} a_n = 4$.

2. (a) Let (a_n) be a sequence of real numbers. State the ϵ-N definition of what it means for (a_n) to converge to $a \in \mathbf{R}$.

A sequence (a_n) of real numbers converges to $a \in \mathbb{R}$ if

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $|a_n - a| < \varepsilon$.

(b) Suppose that the sequence of real numbers (a_n) converges to $a \in \mathbf{R}$. Prove using the above definition that the sequence $(|a_n|)$ converges to $|a|$.

Suppose that (a_n) converges to $a \in \mathbb{R}$, and let $\varepsilon > 0$.

Then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $|a_n - a| < \varepsilon$.

For any $n \in \mathbb{N}$, we have

$\big| |a_n| - |a| \big| \leq |a_n - a|$ by the reverse triangle inequality.

So, $\forall n \geq N$, $\big||a_n| - |a|\big| \leq |a_n - a| < \varepsilon$.

Thus, $(|a_n|)$ converges to $|a|$.

3. (a) State the Mean Value Theorem.

Suppose that $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Then $\exists c\in(a,b)$ such that $f'(c)=\frac{f(b)-f(a)}{b-a}$.

(b) Suppose that $f : A \to \mathbf{R}$ is a real-valued function on a set $A \subseteq \mathbf{R}$. Define what it means for f to be uniformly continuous on A.

$f:A\to\mathbb{R}$ is uniformly continuous on A if $\forall \varepsilon>0\ \exists\delta>0$ such that whenever $x,y\in A$ and $|x-y|<\delta$ we have $|f(x)-f(y)|<\varepsilon$.

(c) Suppose that f is a real-valued function defined on the entire real line. Use the Mean Value Theorem to prove that if $f'(x)$ exists and is bounded on all of $\mathbf{R}$, then f is uniformly continuous on $\mathbf{R}$.

Suppose that $f:\mathbb{R}\to\mathbb{R}$ is differentiable on $\mathbb{R}$ and let $M>0$ be such that $|f'(x)|\le M$ $\forall x\in\mathbb{R}$. Note that since f is differentiable on $\mathbb{R}$, f is continuous on $\mathbb{R}$ as well.

Let $\varepsilon>0$ and set $\delta=\varepsilon/M>0$. Then if $x,y\in\mathbb{R}$ with $0<|x-y|<\delta$, by the MVT there is a c between x and y such that $|f'(c)|=\left|\frac{f(x)-f(y)}{x-y}\right|\le M$. Hence,

$$|f(x)-f(y)|\le M|x-y|<M\delta=\varepsilon.$$

Therefore, f is uniformly continuous on $\mathbb{R}$.

4. (a) State the Weierstrass M-test.

Suppose that $\forall n \in \mathbb{N}$ and $x \in A$, $|f_n(x)| \leq M_n$ and $\sum_{n=1}^{\infty} M_n < \infty$, then $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely & uniformly on A.

(b) Use part (a) to show that for any $r \in (0,1)$ the function $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$ is well-defined and continuous on $[-r, r]$.

Let $r \in (0,1)$ and $f_n(x) = \frac{x^n}{n}$.

Then for any $x \in [-r,r]$ and $n \geq 1$ we have

$$|f_n(x)| = \frac{|x|^n}{n} \leq |x|^n \leq r^n.$$

Since $0 < r < 1$, the geometric series $\sum_{n=1}^{\infty} r^n$ converges.

Thus, by the W. M-test, the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges absolutely and uniformly on $[-r, r]$. Hence, $\forall x \in [-r, r]$, the function $f(x)$ is well-defined.

Moreover, since $S_N(x) = \sum_{n=1}^{N} \frac{x^n}{n}$ is a polynomial, S_N is continuous for all N. Since S_N converges to f uniformly on $[-r,r]$, the limit f is also continuous on $[-r, r]$.