Amherst College
Department of Mathematics and Statistics

Comprehensive Examination

◁ Multivariable Calculus and Linear Algebra ▷

Practice Exam 3

Number: ____

Solutions

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable. Show all your work, and justify your answers.
- The Multivariable Calculus and Linear Algebra Exam consists of Questions 1–8 that total to 200 points.

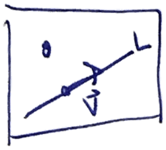
For Department Use Only:

Grader #1: ________________

Grader #2: ________________

1. Find an equation for the plane that passes through the point $(2, 4, 1)$ and contains the line

$$x = 5 - 3t,\ y = 2 + t,\ z = 4 - 6t.$$



Let $\vec{v} = \langle -3, 1, -6 \rangle$ be the direction vector for the line.

So, $\vec{v}$ is parallel to the plane.

Since the plane contains the line, letting $t=0$ gives $(5, 2, 4)$ is another point in the plane. Then $\vec{w} = \langle 5-2, 2-4, 4-1 \rangle = \langle 3, -2, 3 \rangle$ is also parallel to the plane. So, $\vec{n} = \vec{v} \times \vec{w}$ is perpendicular to the plane.

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & -6 \\ 3 & -2 & 3 \end{vmatrix} = \vec{i} \begin{vmatrix} 1 & -6 \\ -2 & 3 \end{vmatrix} - \vec{j} \begin{vmatrix} -3 & -6 \\ 3 & 3 \end{vmatrix} + \vec{k} \begin{vmatrix} -3 & 1 \\ 3 & -2 \end{vmatrix}$$

$$= \langle 3-12, -(-9+18), 6-3 \rangle$$

$$= \langle -9, -9, 3 \rangle$$

So, an equation for the plane is

$$-9(x-2) - 9(y-4) + 3(z-1) = 0$$

$$-3(x-2) - 3(y-4) + (z-1) = 0$$

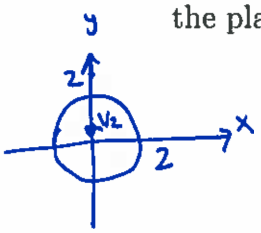
or $\cancel{-10x + 12y + 3z = 20 + 48 + 3}$

$\cancel{+65}$

$$-3x - 3y + z = -(6 + 12 - 1)$$

$$= -17$$

2. A flat circular metal plate has the shape of the disk $x^2 + y^2 \leq 4$. The plate (including the boundary) is heated so that the temperature at a point (x, y) on the plate is given by $T(x, y) = x^2 + 2y^2 - 2y$. Find the temperatures at the hottest and coldest points on the plate and state all points where these extrema occur.



Interior $T_x = 2x = 0 \Rightarrow x = 0$

$T_y = 4y - 2 = 0 \Rightarrow y = 1/2$

Critical point : $(0, 1/2)$ is in the disk and

$T(0, 1/2) = 2(1/4) - 1 = -1/2$

Boundary $x^2 + y^2 = 4$

Let $g(x,y) = x^2 + y^2$. Then $\nabla T = \lambda \nabla g \Rightarrow 2x = \lambda 2x$

and $4y - 2 = \lambda 2y$

Since $2x = \lambda 2x$ either $\lambda = 1$ or $x = 0$.

$\lambda = 1 \Downarrow$: $4y - 2 = 2y$, $2y = 2$, $y = 1$, $x^2 + 1 = 4$, $x = \pm\sqrt{3}$

$x = 0 \Downarrow$: $y^2 = 4$, $y = \pm 2$

$T(\pm\sqrt{3}, 1) = 3 + 2 - 2 = 3$

$T(0, 2) = 8 - 4 = 4$

$T(0, -2) = 8 + 4 = 12$

So, the hottest point on the plate is at $(0, -2)$, where the temperature is 12 and the coldest point is at $(0, 1/2)$, where the temperature is $-1/2$.

3. Let $f(x,y) = \begin{cases} \dfrac{y^2(2x-y)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

(a) Compute $f_x(0,0)$ and $f_y(0,0)$.

$$\lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{0\cdot 2h}{h^2} - 0}{h} = 0 \quad \text{So } f_x(0,0) = 0.$$

$$\lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{(0+h)^2(-h)}{h^2} - 0}{h} = \lim_{h\to 0} \frac{-h^3}{h^3} = -1$$

So $f_y(0,0) = -1$.

(b) Is f continuous at $(0,0)$? Justify your answer.

In polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$f(x,y) = f(r\cos\theta, r\sin\theta) = \frac{r^2\sin^2\theta\,(2r\cos\theta - r\sin\theta)}{r^2} = r\sin^2\theta\,(2\cos\theta - \sin\theta)$$

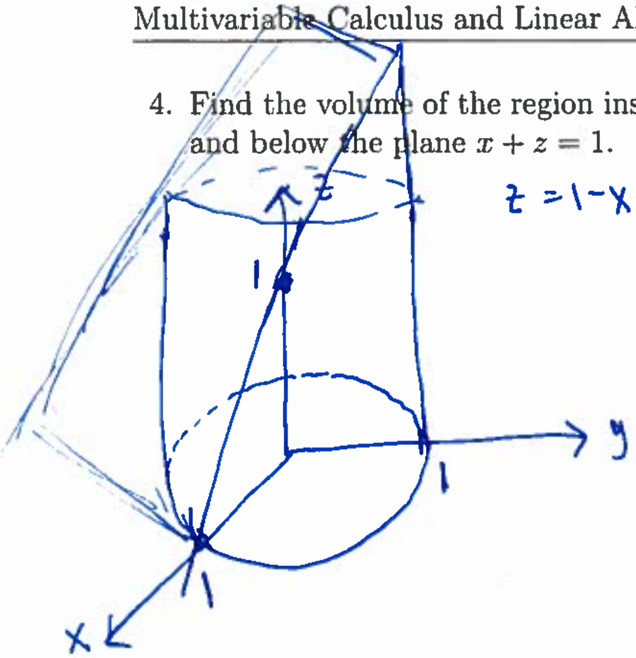
Now $\lim_{r\to 0} f(r\cos\theta, r\sin\theta) = \lim_{r\to 0} r\sin^2\theta\,(2\cos\theta - \sin\theta) = 0.$

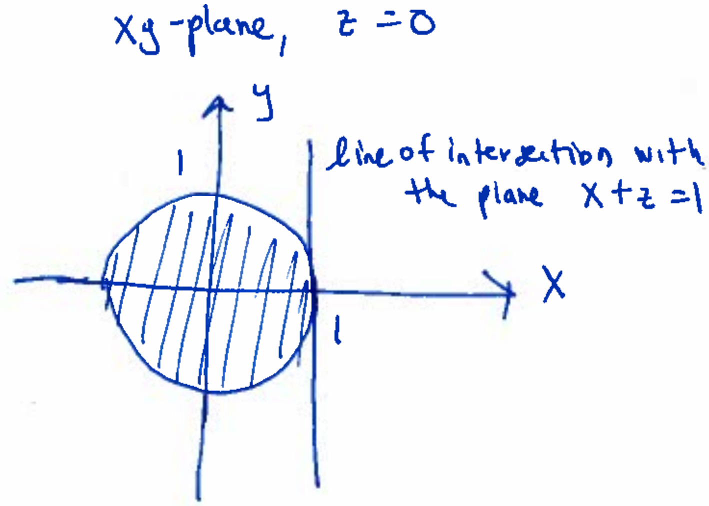
Hence, $\lim_{(x,y)\to(0,0)} f(x,y)$ exists and is equal to zero.

Also, we have $f(0,0) = 0$. Thus, $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$.

So, f is continuous at $(0,0)$.

4. Find the volume of the region inside the cylinder $x^2 + y^2 = 1$, above the xy-plane $z = 0$, and below the plane $x + z = 1$.





$$\text{So, } V = \iint_D (1-x)\, dA$$

$$= \int_0^{2\pi}\int_0^1 (1 - r\cos\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (r - r^2\cos\theta)\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}r^2 - \frac{1}{3}r^3\cos\theta \Big|_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{3}\cos\theta\right) d\theta$$

$$= \frac{1}{2}\theta - \frac{1}{3}\sin\theta \Big|_0^{2\pi}$$

$$= \pi$$

5. (a) Suppose that V is a vector space. Explain what it means to say that a subset W of V is a subspace.

A subset $W \subseteq V$ of a vector space V is a subspace if

$W \neq \emptyset$ and $\forall v, w \in W$, and $c \in \mathbb{R}$, $cv + w \in W$.

(b) Let V_1 and V_2 be vector spaces and let $T : V_1 \to V_2$ be a linear transformation. For $W_2 \subseteq V_2$, let $W_1 = \{v \in V_1 : T(v) \in W_2\}$. Show that if W_2 is a subspace of V_2 then W_1 is a subspace of V_1.

Let W_2 be a subspace of V_2, $T: V_1 \to V_2$ be linear, and $W_1 = \{ v \in V_1 : T(v) \in W_2 \}$. Since T is linear, $T(0_1) = 0_2$, so $0_1 \in W_1$ and $W_1 \neq \emptyset$. Let $x, y \in W_1$ and $c \in \mathbb{R}$. Then $x, y \in V_1$ and $T(x), T(y) \in W_2$. Since W_2 is a subspace of V_2, $cT(x) + T(y) \in W_2$ as well. And

$$\begin{aligned} cT(x) + T(y) &= T(cx) + T(y) \\ &= T(cx + y) \quad \text{since } T \text{ is linear.} \end{aligned}$$

Then $T(cx+y) \in W_2$ and hence $cx + y \in W_1$.

Thus, W_1 is a subspace of V_1.

6. (a) Explain what it means to say that a subset S of a vector space V is a *basis* of V.

A subset $S \subseteq V$ of a vector space V is a basis of V if S in linearly independent and the span of S is equal to V.

(b) Give a basis for the subspace of $\mathbb{R}^3$ spanned by the vectors

Let $v_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -4 \\ 2 \\ -2 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$, $v_4 = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$.

You should explain how you know that your answer really is a basis; namely, you should relate your answer to the definition you gave in part (a) for full credit.

By inspection, we note that $v_2 = -2v_1$ and $v_1 = 2v_3 + v_4$.
so $v_2 = -4v_3 - 2v_4$.

Hence, both v_1 and v_2 are in the span of v_3 and v_4.

Now if $av_3 + bv_4 = 0$ we have $a\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$ $a=0$ and $b=0$.

So, v_3 and v_4 are linearly independent.

Thus, $\{v_3, v_4\}$ is a basis for the subspace spanned by $\{v_1, v_2, v_3, v_4\}$.

7. Suppose that $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map such that $\|T(\mathbf{x})\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$. Prove that T is an isomorphism.

- First we show that T is injective by showing that $\ker(T) = \{0\}$. Let $x \in \ker(T)$. Then $T(x) = 0 \Rightarrow \|T(x)\| = \|x\| = 0$. Thus, ~~xxxx~~ $x = 0$. So, $\ker(T) = \{0\}$, and T is injective as desired.

- Since $\ker(T) = \{0\}$, the nullity of T is 0. Then, since the dimension of $\mathbb{R}^n$ is equal to n, we must have that the rank of T is equal to n, by the rank-nullity theorem. But since the dimension of the image of T is n, which is equal to the dimension of the target space, $\mathbb{R}^n$, T must be surjective.

Thus, T is an isomorphism.

8. Let A be the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

(a) Define what it means for a real number $\lambda \in \mathbb{R}$ to be an eigenvalue of A.

A real number λ is an eigenvalue of A if there exists a nonzero vector v such that $Av = \lambda v$.

(b) Find all eigenvectors of A.

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 1-\lambda & 1 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 2-\lambda \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} - 0 + 0$$

$$= (2-\lambda)\left[(1-\lambda)^2 - 0\right]$$

$$= (2-\lambda)(1-\lambda)^2 = 0 \implies \lambda = 2, 1$$

$\underline{\lambda = 2}$ $\begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{matrix} x-y+z=0 \\ x-z=0 \end{matrix} \implies \begin{matrix} x = z \\ y = 2x \end{matrix}$ Let $x = z = 1$, $y = 2$

$\underline{\lambda = 1}$ $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{matrix} x=0 \\ x+z=0 \end{matrix} \implies x = z = 0$ Let $y = 1$.

Then the eigenvectors of A are the vectors $\begin{bmatrix} x \\ 2x \\ x \end{bmatrix}$ and $\begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix}$

where $x, y \in \mathbb{R} \setminus \{0\}$.

(c) Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$, or show that no such matrices exist.

Since the dimension of the eigenspace corresponding to $\lambda = 1$ is one while the multiplicity of the eigenvalue $\lambda = 1$ is two, we know that A is not diagonalizable. Hence, there are no such matrices D and P.