Amherst College
Department of Mathematics and Statistics

Comprehensive Examination

◁ Algebra ▷

Practice Exam 1

Number: ______

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable. Show all your work, and justify your answers.
- The Algebra Exam consists of Questions 1–4 that total to 100 points.

For Department Use Only:

Grader #1: ____________________

Grader #2: ____________________

1. Let G_1 and G_2 be groups and $\phi : G_1 \to G_2$ be a homomorphism. Suppose that N_2 is a subgroup of G_2 and define the set

$$N_1 = \{a \in G_1 : \phi(a) \in N_2\}.$$

   (a) Prove that N_1 is a subgroup of G_1.
   [Note: this is a standard theorem in Math 350. Since you are being asked to prove that theorem here, you may not quote that theorem.]

   (b) Prove that if N_2 is a normal subgroup of G_2 then N_1 is a normal subgroup of G_1.

2. Let G be a finite group. Suppose that x and y are distinct elements of order two in G such that $xy = yx$. Prove that the order of G is divisible by 4.

3. Suppose that σ is a permutation in the **alternating group** A_{10} given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 7 & 2 & 6 & 10 & 1 & 5 & & & 3 \end{pmatrix}$$

where the images of 8 and 9 have been lost.

(a) Determine the images of 8 and 9 under σ. Don't forget to justify your answer.

(b) Compute the order of σ.

4. Let R be a ring.

   (a) Define what it means for a subset $I \subseteq R$ to be an **ideal** of R.

   If you use any other technical terms like "closed," "subring," "group," "subgroup," etc., you must fully define those terms as well.

   (b) Let $R = \mathbb{R}[x]$ be the ring of polynomials with coefficients in the field $\mathbb{R}$ of real numbers. Let $I \subseteq R$ be the subset

   $$I = \{f \in \mathbb{R}[x] : f(1) = f(2) = 0\}.$$

   Prove that I is an ideal of R.

   (c) With $R = \mathbb{R}[x]$ and I defined in part (b), prove that R/I has zero-divisors. That is, show that there are two nonzero elements of R/I whose product is zero in R/I.