Amherst College
Department of Mathematics and Statistics

Comprehensive Examination

◁ Analysis ▷

Practice Exam 1

Number: ______

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable. Show all your work, and justify your answers.
- The Analysis Exam consists of Questions 1–4 that total to 100 points.

For Department Use Only:

Grader #1: __________________

Grader #2: __________________

1. (a) State the Axiom of Completeness.

   (b) Let (a_n) be a sequence of real numbers. State the ϵ-N definition of what it means for (a_n) to converge to $a \in \mathbf{R}$.

   (c) Let (a_n) be an increasing sequence of real numbers and suppose that there exists a real number $M \in \mathbf{R}$ such that $a_n \leq M$ for all n. Use the Axiom of Completeness and the definition in part (b) to prove that the sequence (a_n) converges.

2. (a) Let $f\colon A \to \mathbf{R}$ be a function. Using the ϵ-δ definition, define what it means for f to be continuous at $c \in \mathbf{A}$.

(b) Suppose that the functions $f, g\colon \mathbf{A} \to \mathbf{R}$ are both continuous at $c \in A$. Prove using the above definition that the function $h : A \to \mathbb{R}$ defined by $h(x) = f(x) + g(x)$ is continuous at c.

3. Suppose that we have a collection of compact sets $K_\lambda \subset \mathbf{R}$ for all λ in some index set Λ.
   (a) Give a condition that is both necessary and sufficient for a set of real numbers to be compact in $\mathbf{R}$.

   (b) Use the condition in part (a) to prove that the intersection $K = \bigcap_{\lambda \in \Lambda} K_\lambda$ is compact.

   (c) Give an example to show that the union $\bigcup_{\lambda \in \Lambda} K_\lambda$ is not necessarily compact.

4. Consider the sequence of functions (f_n) where $f_n(x) = \frac{1}{1+n^2x^2}$ for $n \geq 1$.
   (a) Prove that (f_n) converges pointwise to a function f on $[0, 1]$.

   (b) Prove that (f_n) does not converge uniformly on $[0, 1]$.