Amherst College
Department of Mathematics and Statistics

Comprehensive Examination

◁ Multivariable Calculus and Linear Algebra ▷

Practice Exam 1

Number: ______

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable. Show all your work, and justify your answers.
- The Multivariable Calculus and Linear Algebra Exam consists of Questions 1–8 that total to 200 points.

For Department Use Only:

Grader #1: __________________

Grader #2: __________________

1. Consider the surface S given by $x^2y - yz^2 + z = 1$.

   (a) Find an equation of the tangent plane to S at the point $(11, 0, 1)$.

   (b) Find two points on the surface S where the tangent plane is parallel to the yz-plane.

2. Let $f(x, y) = 3x^2 - 3xy + y^3$. Find all critical points of f, and classify each as a local maximum, local minimum, or saddle point.

3. Let $f(x,y) = \begin{cases} \dfrac{3x^2y^2}{2x^4+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

(a) Compute $f_x(0,0)$ and $f_y(0,0)$.

(b) ~~Is f continuous at $(0,0)$? Justify your answer.~~ *This is no longer on the Comps syllabus.*

4. Compute $\int_C y^2\ dx + 3xy\ dy$ where C is the boundary curve of the region bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ in the upper half plane $y \geq 0$, traversed in the counterclockwise direction.
   Note. This integral may also be written as $\int_C \langle y^2, 3xy \rangle \cdot d\boldsymbol{r}$

5. (a) Let U and V be subspaces of a vector space W. Prove that $U + V = \{u + v : u \in U, v \in V\}$ is a subspace of W.

(b) Suppose $\{u_1, \ldots u_m\}$ is a basis for U and $\{v_1, \ldots v_n\}$ is a basis for V. Prove that $\{u_1, \ldots, u_m, v_1, \ldots v_n\}$ spans $U + V$.

(c) Prove that $\dim(U + V) \leq \dim(U) + \dim(V)$.

6. Let V be a vector space.

   (a) Explain what it means to say that a subset S of V is a *basis* of V.

   (b) Suppose that $\{u, v, w\}$ is a basis of V. Prove that $\{u + 2v, v + 2w, u + 2w\}$ is also a basis of V.

7. Consider the matrix $A = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$.

   (a) Find all eigenvalues of A.

   (b) Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$, or show that no such matrices exist.

8. Let $P_2 = \{a + bt + ct^2 : a, b, c \in \mathbb{R}\}$ and $T : P_2 \to \mathbb{R}^2$ be defined by

$$T(p) = \begin{bmatrix} p(1) \\ p(2) \end{bmatrix}.$$

(a) Prove that T is linear.

(b) Find the matrix representation of T with respect to the bases $\{1, t, t^2\}$ and $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

(c) Find the rank and nullity of T.