Amherst College
Department of Mathematics and Statistics

Comprehensive Examination

◁ Algebra ▷

Practice Exam 2

Number: _____

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable. Show all your work, and justify your answers.
- The Algebra Exam consists of Questions 1–4 that total to 100 points.

For Department Use Only:

Grader #1: ________________

Grader #2: ________________

1. Let G_1 and G_2 be finite groups and $\phi : G_1 \to G_2$ be a homomorphism. Suppose that $x \in G_1$ has order $n \geq 1$.

   (a) Show that the order of $\phi(x)$ divides n.

   (b) Prove that if the order of G_2 is relatively prime to n, then x is in the kernel of ϕ.

2. Let G be a group and define $Z = \{g \in G : ga = ag \text{ for all } a \in G\}$.
   (a) Show that Z is a subgroup of G.

   (b) Show that the subgroup Z is normal in G.

   (c) Prove that if the quotient group G/Z is cyclic, then G is abelian.

3. Consider the group S_9 of permutations of the set $\{1, 2, 3, \ldots, 9\}$. Let $\sigma, \tau \in S_9$ be the permutations

$$\sigma = (1,4,3)(9,5,7) \quad \text{and} \quad \tau = (3,9)(1,5,8).$$

(a) Write $\tau\sigma^2$ as a product of **disjoint** cycles.

(b) Compute the **order** of each of σ, τ, and $\tau\sigma^2$.

(c) Decide whether each of σ, τ, and $\tau\sigma^2$ is an **even** or **odd** permutation; don't forget to justify.

4. Let R be a ring.

   (a) Define what it means for a subset $I \subseteq R$ to be an **ideal** of R.

   If you use any other technical terms like "closed," "subring," "group," "subgroup," etc., you must fully define those terms as well.

   (b) Let $I \subseteq R$ be an ideal of R, S be another ring, and $\phi : R \to S$ be a ring homomorphism. Prove that if ϕ is surjective then $\phi(I) = \{\phi(x) : x \in I\}$ is an ideal of S.