Amherst College
Department of Mathematics and Statistics

Comprehensive Examination

◁ Analysis ▷

Practice Exam 2

Number: ______

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable. Show all your work, and justify your answers.
- The Analysis Exam consists of Questions 1–4 that total to 100 points.

For Department Use Only:

Grader #1: ____________________

Grader #2: ____________________

1. Let $S_N = \displaystyle\sum_{k=1}^{N} \frac{1}{k}$.

   (a) Use induction to prove that $S_{2^N} \geq \frac{1}{2}(N+2)$ for every $N \geq 0$.

   (b) Use part (a) to prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge.

2. (a) Let (a_n) be a sequence of real numbers. State the ϵ-N definition of what it means for (a_n) to converge to $a \in \mathbf{R}$.

(b) State the Bolzano-Weierstrass Theorem.

(c) Let (a_n) be a Cauchy sequence in $\mathbf{R}$. Use the definition in part (a) and the Bolzano-Weierstrass Theorem from part (b) to prove that (a_n) converges.

3. (a) State the Intermediate Value Theorem.

(b) Suppose $f : [-1, 1] \to \mathbf{R}$ is continuous and satisfies $f(-1) = f(1)$. Use the Intermediate Value Theorem to prove that there exists a number $\gamma \in [0, 1]$ such that $f(\gamma) = f(\gamma - 1)$.

4. (a) Let (f_n) be a sequence of bounded functions on $[a, b]$. Prove that if (f_n) converges uniformly to f on $[a, b]$, then f is bounded.

   (b) Give an example to show that the statement in part (a) is false if uniform convergence is replaced by pointwise convergence.