Amherst College
Department of Mathematics and Statistics

Comprehensive Examination

◁ Multivariable Calculus and Linear Algebra ▷

Practice Exam 2

Number: ______

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable. Show all your work, and justify your answers.
- The Multivariable Calculus and Linear Algebra Exam consists of Questions 1–8 that total to 200 points.

For Department Use Only:

Grader #1: __________________

Grader #2: __________________

1. Let $F(x, y, z) = x^2 + xy^2 + z$.

   (a) Find an equation of the tangent plane to the surface $F(x, y, z) = 4$ at the point $(1, 2, -1)$.

   (b) Find the directional derivative of F at the point $(1, 2, -1)$. in the direction of the tangent vector to the curve $\vec{r}(t) = \langle 2t^2 - t, t, t^2 - 2t^3 \rangle$ at $t = 1$.

2. Find the points at which the absolute maximum and minimum values of the function $f(x, y) = x^2 + 2y^2 + 5$ on the region $x^2 + 4y^2 \leq 4$ occur. State all points where the extrema occur as well as the maximum and minimum values.

3. Calculate the volume of the region that lies both inside the sphere $x^2 + y^2 + z^2 = 9$ and above the cone $z = \sqrt{x^2 + y^2}$.

4. Compute $\int_C 3x^2y\,dx + (x^3 + e^y)\,dy$ where C is the circle $x^2 + y^2 + x = 1$, traversed in the counterclockwise direction.
Note. This integral may also be written as $\int_C \langle 3x^2y, x^3 + e^y\rangle \cdot d\boldsymbol{r}$

5. Let $M_n(\mathbb{R})$ be the vector space of all $n \times n$ matrices with real coefficients. We say that $A, B \in M_n(\mathbb{R})$ commute if $AB = BA$.

   (a) Fix $A \in M_n(\mathbb{R})$. Prove that the set of all matrices in $M_n(\mathbb{R})$ that commute with A is a subspace of $M_n(\mathbb{R})$.

   (b) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in M_2(\mathbb{R})$ and let $W \subseteq M_2(\mathbb{R})$ be the subspace of all matrices in $M_2(\mathbb{R})$ that commute with A. Find a basis of W.

6. Let V and W be vector spaces and let T be a linear transformation from V to W.

   (a) Prove that the kernel of T (also called the null space of T) is a subspace of V.

   (b) Prove that T is one-to-one if and only if the kernel of T is $\{0\}$.

   (c) Suppose that T is one-to-one and $\{v_1, v_2, \ldots v_n\}$ is a set of n linearly independent vectors in V. Prove that $\{T(v_1), T(v_2), \ldots T(v_n)\}$ is linearly independent in W.

7. Let P_n be the vector space of polynomials in x with real coefficients of degree at most n. Define $T : P_2 \to P_3$ by $T(f) = \int_0^x f(t)\,dt$. You may assume that T is linear.

   (a) Compute $T(a + bx + cx^2)$ where $a, b, c \in \mathbb{R}$.

   (b) Compute the matrix of T with respect to the bases $\{1, x, x^2\}$ of P_2 and $\{1, x, x^2, x^3\}$ of P_3.

8. (a) Let V be a vector space and $T : V \to V$ and $U : V \to V$ be linear transformations that commute, i.e. $T \circ U = U \circ T$. Let $v \in V$ be an eigenvector of T such that $U(v) \neq 0$. Prove that $U(v)$ is also an eigenvector of T.

(b) Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation that has 0 as an eigenvalue. What are the possible values of the rank of T? Justify your answer.