Amherst College
Department of Mathematics and Statistics

Comprehensive Examination

◁ Algebra ▷

Practice Exam 3

Number: ______

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable. Show all your work, and justify your answers.
- The Algebra Exam consists of Questions 1–4 that total to 100 points.

For Department Use Only:

Grader #1: ____________________

Grader #2: ____________________

1. Let G be a group and let $I(G) = \{x \in G : x = x^{-1}\}$.
   (a) Show that if G is abelian, then $I(G)$ is a subgroup of G.

   (b) Show that if G is finite and $I(G) \neq \{e\}$, then G must have even order.

   (c) Give an example of a group G for which $I(G)$ is not a subgroup of G.

2. Let G be a group.

   (a) Let $g, h \in G$. Prove that for all integers $k \geq 1$,

   $$(h^{-1}gh)^k = h^{-1}g^k h.$$

   (b) Let N be a normal subgroup of G and suppose that N is cyclic. Let H be a subgroup of N. Prove that H is a normal subgroup of G.

3. Consider the group S_{12} of permutations of the set $\{1, 2, 3, \ldots, 12\}$.
   (a) Give an example of an even permutation $\sigma \in S_{12}$. Don't forget to justify your answer.

   (b) Give an example of a permutation $\tau \in S_{12}$ that has order 14. Don't forget to justify your answer.

   (c) Prove that no element of S_{12} has order 13.

4. Let R be a ring.

   (a) Define what it means for a subset $I \subseteq R$ to be an **ideal** of R.

   If you use any other technical terms like "closed," "subring," "group," "subgroup," etc., you must fully define those terms as well.

   (b) Let $I \subseteq R$ be an ideal of R, and suppose that $xy - yx \in I$ for every $x, y \in R$. Prove that the quotient ring R/I is commutative.