Amherst College
Department of Mathematics and Statistics

Comprehensive Examination

◁ Analysis ▷

Practice Exam 3

Number: _____

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable. Show all your work, and justify your answers.
- The Analysis Exam consists of Questions 1–4 that total to 100 points.

For Department Use Only:

Grader #1: ________________

Grader #2: ________________

1. Consider the sequence (a_n) defined recursively as follows.

$$a_1 = 2 \qquad \text{and} \qquad a_{n+1} = 5 - \frac{4}{a_n} \qquad \text{for} \qquad n \geq 1.$$

(a) Prove that for $n \geq 1$, $a_{n+1} \geq a_n$.

(b) Prove that the sequence (a_n) is bounded from above.

(c) Prove that the sequence (a_n) converges and find $\lim_{n\to\infty} a_n$.

2. (a) Let (a_n) be a sequence of real numbers. State the ϵ-N definition of what it means for (a_n) to converge to $a \in \mathbf{R}$.

(b) Suppose that the sequence of real numbers (a_n) converges to $a \in \mathbf{R}$. Prove using the above definition that the sequence $(|a_n|)$ converges to $|a|$.

3. (a) State the Mean Value Theorem.

(b) Suppose that $f : A \to \mathbf{R}$ is a real-valued function on a set $A \subseteq \mathbf{R}$. Define what it means for f to be uniformly continuous on A.

(c) Suppose that f is a real-valued function defined on the entire real line. Use the Mean Value Theorem to prove that if $f'(x)$ exists and is bounded on all of $\mathbf{R}$, then f is uniformly continuous on $\mathbf{R}$.

4. (a) State the Weierstrass M-test.

(b) Use part (a) to show that for any $r \in (0,1)$ the function $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$ is well-defined and continuous on $[-r, r]$.