Amherst College
Department of Mathematics and Statistics

Comprehensive Examination
◁ Multivariable Calculus and Linear Algebra ▷
Practice Exam 3

Number: _____

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable. Show all your work, and justify your answers.
- The Multivariable Calculus and Linear Algebra Exam consists of Questions 1–8 that total to 200 points.

For Department Use Only:

Grader #1: _________________

Grader #2: _________________

1. Find an equation for the plane that passes through the point $(2, 4, 1)$ and contains the line
$$x = 5 - 3t,\ y = 2 + t,\ z = 4 - 6t.$$

2. A flat circular metal plate has the shape of the disk $x^2 + y^2 \leq 4$. The plate (including the boundary) is heated so that the temperature at a point (x, y) on the plate is given by $T(x, y) = x^2 + 2y^2 - 2y$. Find the temperatures at the hottest and coldest points on the plate and state all points where these extrema occur.

3. Let $f(x,y) = \begin{cases} \dfrac{y^2(2x-y)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

(a) Compute $f_x(0,0)$ and $f_y(0,0)$.

(b) ~~Is f continuous at $(0,0)$? Justify your answer.~~ *This is no longer on the Comps syllabus.*

4. Find the volume of the region inside the cylinder $x^2 + y^2 = 1$, above the xy-plane $z = 0$, and below the plane $x + z = 1$.

5. (a) Suppose that V is a vector space. Explain what it means to say that a subset W of V is a subspace.

(b) Let V_1 and V_2 be vector spaces and let $T : V_1 \to V_2$ be a linear transformation. For $W_2 \subseteq V_2$, let $W_1 = \{v \in V_1 : T(v) \in W_2\}$. Show that if W_2 is a subspace of V_2 then W_1 is a subspace of V_1.

6. (a) Explain what it means to say that a subset S of a vector space V is a *basis* of V.

(b) Give a basis for the subspace of $\mathbb{R}^3$ spanned by the vectors

$$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -4 \\ 2 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}.$$

You should explain how you know that your answer really is a basis; namely, you should relate your answer to the definition you gave in part (a) for full credit.

7. Suppose that $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map such that $\|T(\mathbf{x})\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$. Prove that T is an isomorphism.

8. Let A be the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

   (a) Define what it means for a real number $\lambda \in \mathbb{R}$ to be an eigenvalue of A.

   (b) Find all eigenvectors of A.

   (c) Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$, or show that no such matrices exist.