Math 13 Spring 2020: Exam 2

March 30, 2010

Name:

Instructions: There are 4 questions on this exam each scored out of 8 points for a total of 32 points. You may not use any outside materials(eg. notes or calculators). You have 50 minutes to complete this exam. Remember to fully justify your answers.

Score:

Problem 1. Evaluate the following limits or show that they do not exist.

(a) $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x+y^2}$

(b) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

Proof.

(a) We get

$$\left|\frac{3x^2y}{x+y^2}\right| \le |3xy|$$

Additionally,

$$\lim_{(x,y)\to(0,0)} |3xy| = 0.$$

Therefore by the squeeze theorem

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x+y^2} = 0.$$

(b) We consider paths of the form $y = mx$ to get

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{mx^2}{x^2(1+m^2)} = \frac{m}{1+m^2}.$$

Since this depends on m, different paths gives different values, so the limit does not exist.

□

Problem 2. Given that $z = f(x^2 + y^2, x^2 - y^2)$.

(a) Compute z_{xy}.

(b) Approximate z at $(x, y) = (1.1, 0.9)$ if $f(2, 0) = 7$, $f_s(2, 0) = 1$, and $f_t(2, 0) = -2$ where $s = x^2 + y^2$ and $t = x^2 - y^2$.

Proof.

(a) We set $s = x^2 + y^2$ and $t = x^2 - y^2$ and use the chain rule to compute

$$z_x = 2x(f_s + f_t)$$

and

$$z_{xy} = 2x(f_{ss}2y + f_{st}(-2y) + f_{ts}(2y) + f_{tt}(-2y)).$$

(b) We make the approximation

$$\begin{aligned} z &= z_0 + z_x \Delta x + z_y \Delta y \\ &= 7 + 2x(f_s + f_t)\frac{1}{10} + 2y(f_s - f_t)\left(-\frac{1}{10}\right) \\ &= 7 + 2(1-2)\frac{1}{10} + 2(1+2)\left(-\frac{1}{10}\right) \\ &= 7 - \frac{2}{10} - \frac{6}{10} = 6\frac{1}{5}. \end{aligned}$$

or we could approximate

$$\begin{aligned} z &= z_0 + z_s \Delta s + z_t \Delta t \\ &= 7 + 1(1.1^2 + 0.9^2 - 2) - 2(1.1^2 - 0.9^2) \\ &= 7 + 0.02 - 0.8 = 6.22. \end{aligned}$$

□

Problem 3. Find and classify the critical points of $f(x,y) = xy - \frac{x^4}{4} - \frac{y^4}{4}$.

Proof. We have

$$\begin{aligned} f_x &= y - x^3 \\ f_y &= x - y^3 \end{aligned}$$

There are three critical points $(0,0)$, $(-1,-1)$, and $(1,1)$. The second derivatives are

$$\begin{aligned} f_{xx} &= -3x^2 \\ f_{xy} &= 1 \\ f_{yy} &= -3y^2 \end{aligned}$$

So we have

$$\begin{aligned} D(-1,-1) &= 9 - 1 = 8 \quad f_{xx}(-1,-1) = -3 \\ D(1,1) &= 9 - 1 = 8 \quad f_{xx}(1,1) = -3 \\ D(0,0) &= -1 \end{aligned}$$

By the second derivative test we have $(-1,-1)$ and $(1,1)$ are relative maximums and $(0,0)$ is a saddle point. □

Problem 4. Find the maximum and minimum values of $f(x, y) = 4x^2 + 10y^2$ on the disk $x^2 + y^2 \leq 4$.

Proof. We take first derivatives

$$f_x = 8x \quad \text{and} \quad f_y = 20y$$

to get the critical point $(0, 0)$ which is in the region.

Using Lagrange multiplier for $f(x, y)$ with the constraint $x^2 + y^2 = 4$ we solve the system

$$\begin{aligned} 8x &= \lambda 2x \\ 20y &= \lambda 2y \\ x^2 + y^2 &= 4. \end{aligned}$$

The first equation implies $x = 0$ or $\lambda = 4$. If $x = 0$ we have the two points $(0, \pm 2)$. If $\lambda = 4$, then the second equation implies $y = 0$. Thus we have two more points $(\pm 2, 0)$.

To determine the max and min values we evaluate the function at these 5 points to get: $f(0, 0) = 0$ for the min, $f(0, \pm 2) = 40$ for the max. □