# Math 12: Final Exam

**Name:**

**Instructions:** There are 8 questions on this exam for a total of 100 points. You may not use any outside materials (eg. notes, calculators, cell phones, etc.). You have 3 hours to complete this exam. Remember to fully justify your answers.

**Problem 1** (12 Points)**.** Find the following limits:

(a) $\lim_{x\to 3} \frac{\ln x - 2}{x^2 - 3x}$

(b) $\lim_{x\to 0} \frac{\sinh x}{x}$

(c) $\lim_{x\to 0} x \cot x$

*Proof.*

(a) This is of the form $\frac{0}{0}$ so we apply l'Hopital's rule to get

$$\lim_{x\to 3} \frac{\ln x - 2}{x^2 - 3x} = \lim_{x\to 1} \frac{1/(x-2)}{2x-3} = \frac{1}{3}.$$

(b) This is of the form $\frac{0}{0}$ so we apply l'Hopital's rule to get

$$\lim_{x\to 0} \frac{\sinh x}{x} = \lim_{x\to 0} \frac{1/2(e^x - e^{-x})}{x} = \lim_{x\to 0} \frac{1/2(e^x + e^{-x})}{1} = 1.$$

(c) This is of the form $0 \cdot \infty$ so we apply l'Hopital's rule to the fraction

$$\lim_{x\to 0} x \cot x = \lim_{x\to 0} \frac{x}{\tan x} = \lim_{x\to 0} \frac{1}{\sec^2 x} = 1.$$

□

**Problem 2** (12 Points)**.** Evaluate the following integrals:

(a) $\displaystyle\int \frac{x+3}{\sqrt{9-x^2}}\, dx.$

(b) $\displaystyle\int \frac{dx}{x^3 + x^2 - 2x}.$

(c) $\displaystyle\int x \sec^2 x \, dx.$

*Proof.*

(a) We make the trig substitution $x = 3\sin\theta$

$$\begin{aligned}\int \frac{x+3}{\sqrt{9-x^2}}dx &= \int \frac{3\sin\theta+3}{\sqrt{9-9\sin^2\theta}} 3\cos\theta d\theta \\ &= \int 3\sin\theta + 3d\theta \\ &= -3\cos\theta + 3\theta + C \\ &= -\sqrt{9-x^2} + 3\arcsin\frac{x}{3} + C.\end{aligned}$$

(b) We factor and use partial fractions to see

$$\frac{1}{x^3+x^2-2x} = \frac{1}{x(x+2)(x-1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-1}$$

and so we have to solve

$$A(x+2)(x-1) + Bx(x-1) + Cx(x+2) = 1.$$

So we choose $x = 0$ to get $A = -\frac{1}{2}$, $x = -2$ to get $B = \frac{1}{6}$, and $x = 1$ to get $C = \frac{1}{3}$. So we integrate

$$\begin{aligned}\int \frac{dx}{x^3+x^2-2x}dx &= \int \frac{-1/2}{x} + \frac{1/6}{x+2} + \frac{1/3}{x-1}dx \\ &= -\frac{1}{2}\ln|x| + \frac{1}{6}\ln|x+2| + \frac{1}{3}\ln|x-1| + C.\end{aligned}$$

(c) Applying integration by parts with $u = x$ and $dv = \sec^2 x dx$ we get

$$\int x\sec^2 x dx = x\tan x - \int \tan x dx = x\tan x + \ln|\cos x| + C.$$

□

**Problem 3** (8 Points)**.** For each of the following improper integrals, determine whether it converges or diverges, and if it converges, find its value.

(a) $\displaystyle\int_1^\infty \frac{dx}{x^2-2x+5}$.

(b) $\displaystyle\int_0^9 \frac{dx}{(x-1)^{4/3}}$.

*Proof.*

(a) This is an improper integral and we need to complete the square in the denominator to get

$$\int_1^\infty \frac{dx}{x^2-2x+5} = \lim_{c\to\infty}\int_1^c \frac{dx}{(x-1)^2+4}$$

We notice that this is of the form $\frac{1}{u^2+a^2}$ with $a = 2$ and $u = x - 1$ so we get

$$\lim_{c\to\infty} \int_1^c \frac{dx}{(x-1)^2+4} = \lim_{c\to\infty} \left( \frac{1}{2} \arctan \frac{x-1}{2} \right)_1^c$$
$$= \lim_{c\to\infty} \frac{1}{2} \arctan \frac{c-1}{2} = \frac{1}{2}\frac{\pi}{2} = \frac{\pi}{4}.$$

So this integral converges and has value $\frac{\pi}{4}$.

(b) Notice that there is a discontinuity at $x = 1$ so we have

$$\int_0^9 \frac{dx}{(x-1)^{4/3}} = \lim_{c\to 1} \int_0^c \frac{dx}{(x-1)^{4/3}} + \lim_{d\to 1} \int_d^9 \frac{dx}{(x-1)^{4/3}}$$

Since

$$\int \frac{dx}{(x-1)^{4/3}} = -3(x-1)^{-1/3}$$

we get

$$\int_0^9 \frac{dx}{(x-1)^{4/3}} = \left( \lim_{c\to 1} -3(c-1)^{-1/3} - 3 \right) + \left( \lim_{d\to 1} -3(d-1)^{-1/3} + \frac{3}{2} \right) = \infty$$

so this integral diverges.

□

**Problem 4** (8 Points)**.** Let $R$ be the region bounded by the curves $y = x^2$ and $y = x + 2$.

(a) Set up (but *don't* evaluate) an integral for the volume of the solid obtained by rotating $R$ about the $x$-axis.

(b) Set up (but *don't* evaluate) an integral for the volume of the solid obtained by rotating $R$ about the line $x = 2$.

*Proof.* The intersection points of the curve and the line are $x = -1$ and $x = 2$.

(a) The cross sections are washers so we have

$$V = \pi \int_{-1}^2 (x+2)^2 - (x^2)^2 dx.$$

(b) The cross sections are cylindrical shells so we have

$$V = 2\pi \int (2-x)(x+2-x^2)dx.$$

□

**Problem 5** (10 Points)**.** Consider the curve given by $x = \sin^3 t$ and $y = \cos^3 t$ from $t = 0$ to $t = \frac{\pi}{2}$.

(a) Find the tangent line(s) to the curve at $\left(\frac{3\sqrt{3}}{8}, \frac{1}{8}\right)$.

(b) Find the length of the curve.

*Proof.* We compute

$$x' = 3\sin^2 t \cos t$$
$$y' = -3\cos^2 t \sin t.$$

(a) The $t$ value for this point occurs where $\cos t = \frac{1}{2}$ and so $t = \frac{\pi}{3}$. So the tangent line has slope

$$\frac{dy}{dx} = \frac{y'}{x'} = -\frac{\cos t}{\sin t} = -\frac{1}{\sqrt{3}}.$$

Then the line is given by

$$y - \frac{1}{8} = -\frac{1}{\sqrt{3}}\left(x - \frac{3\sqrt{3}}{8}\right)$$

which simplifies to

$$y = -\frac{1}{\sqrt{3}}x + \frac{1}{2}.$$

(b) For the arclength we have

$$\begin{aligned} s &= \int_0^{\pi/2} \sqrt{9\sin^4 t\cos^2 t + 9\cos^4 t\sin^2 t}dt \\ &= \int_0^{\pi/2} \sqrt{9\sin^2 t\cos^2 t(\sin^2 t + \cos^2 t)}dt \\ &= \int_0^{\pi/2} 3\sin t\cos t dt \\ &= \left(\frac{3}{2}\sin^2 t\right)_0^{\pi/2} = \frac{3}{2}. \end{aligned}$$

□

**Problem 6** (6 Points)**.** Let $C_1$ be the curve given by the polar coordinates equation $r = 2\sin\theta$, $0 \le \theta \le \pi$, and let $C_2$ be the curve given by the polar coordinates equation $r = 1$. Find the area of the region inside $C_1$ and outside $C_2$.

*Proof.* $r = 1$ is a circle of radius 1 centered at $(0, 0)$ and $r = 2\sin\theta$ is a circle of radius 1 centered at $(0, 1)$. We find the points of intersection as

$$2\sin\theta = 1$$

and so $\theta = \{\frac{\pi}{4}, \frac{3\pi}{4}\}$. So we have for the area

$$\begin{aligned}
A &= \int_{\pi/4}^{3\pi/4} \frac{1}{2}(4\sin^2\theta - 1)d\theta \\
&= \int_{\pi/4}^{3\pi/4} 2\left(\frac{1}{2}(1-\cos 2\theta)\right) - \frac{1}{2}d\theta \\
&= \int_{\pi/4}^{3\pi/4} \frac{1}{2} - \cos 2\theta d\theta \\
&= \left(\frac{\theta}{2} - \frac{\sin 2\theta}{2}\right)_{\pi/4}^{3\pi/4} \\
&= \frac{3\pi}{8} + \frac{1}{2} - \frac{\pi}{8} + \frac{1}{2} \\
&= \frac{\pi}{4} + 1.
\end{aligned}$$

□

**Problem 7** (6 Points)**.** Find the area of the surface obtained when the curve $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$ for $1 \leq x \leq 2$ is rotated about the $y$-axis.

*Proof.* We compute

$$y' = \frac{1}{2}x^2 - \frac{1}{2x^2}$$

and so

$$1 + (y')^2 = \frac{1}{4}x^4 + \frac{1}{2} + \frac{1}{4x^4} = \left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right)^2$$

So we have

$$\begin{aligned}
S &= 2\pi \int_1^2 x\left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right)dx = 2\pi\left(\frac{1}{8}x^4 + \frac{1}{2}\ln|x|\right)_1^2 \\
&= 2\pi\left(2 + \ln 2 - \frac{1}{8} - 0\right) = 2\pi\left(\frac{7}{8} + \ln 2\right).
\end{aligned}$$

□

**Problem 8** (12 Points)**.** Determine whether each series converges absolutely, converges conditionally, or diverges. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n+10)}{n^2+10n}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{2^n n^2}$.

(c) $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{\sqrt{n}}{n+2}$.

*Proof.*

(a) We know that $|\cos x| \leq 1$ and so

$$\sum_{n=1}^{\infty}\left|\frac{\cos(n+10)}{n^2+10n}\right| \leq \sum_{n=1}^{\infty}\frac{1}{n^2+10n} \leq \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

The last sequence is a $p$-series with $n = 2$ and so is convergent, so by the comparison test $\sum_{n=1}^{\infty}\left|\frac{\cos(n+10)}{n^2+10n}\right|$ converges and hence $\sum_{n=1}^{\infty}\frac{\cos(n+10)}{n^2+10n}$ converges absolutely.

(b) Applying the ratio test we see

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{(n+1)!}{2^{n+1}(n+1)^2}\cdot\frac{2^n n^2}{n!} \\ &= \lim_{n\to\infty}\frac{(n+1)n^2}{2(n+1)^2} \\ &= \lim_{n\to\infty}\frac{1}{2}\frac{n^2}{n+1} = \infty\end{aligned}$$

So the series diverges.

(c) Applying the alternating series test we have that the series is decreasing since

$$a_{n+1} = \frac{\sqrt{n+1}}{n+3} < \frac{\sqrt{n}}{n+2} = a_n$$

and that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\sqrt{n}}{n+2} = 0$$

so the series is convergent. However, using the limit comparison test with the divergent series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ with the series $\sum_{n=0}^{\infty}\left|(-1)^n\frac{\sqrt{n}}{n+2}\right| = \sum_{n=0}^{\infty}\frac{\sqrt{n}}{n+2}$ we get

$$\lim_{n\to\infty}\frac{\sqrt{n}}{n+2}\cdot\sqrt{n} = \lim_{n\to\infty}\frac{n}{n+2} = 1$$

and so both series diverge and hence $\sum_{n=0}^{\infty}\left|(-1)^n\frac{\sqrt{n}}{n+2}\right|$ is conditionally convergent.

□

**Problem 9** (8 Points)**.** Find the interval of convergence of the power series $\displaystyle\sum_{n=2}^{\infty}\frac{(x+2)^n}{2^n \ln n}$.

*Proof.* We apply the ratio test:

$$\lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{2^{n+1}\ln(n+1)}\cdot\frac{2^n\ln n}{(x+2)^n}\right| = \lim_{n\to\infty}|x+2|\left|\frac{\ln(n+1)}{\ln n}\right|.$$

We apply L'Hopital's Rule to get

$$\lim_{n\to\infty}|x+2|\left|\frac{\ln(n+1)}{\ln n}\right| = \lim_{n\to\infty}\left|\frac{x+2}{2}\right|\left|\frac{n}{n+1}\right| = \left|\frac{x+2}{2}\right|.$$

So we have $R = 2$ and convergence for $-4 < x < 0$. We now check the end points.

For $x = 0$ we have the series

$$\sum_{n=2}^{\infty}\frac{1}{\ln n}$$

and we know $\frac{1}{\ln n} > \frac{1}{n}$ for all $n \geq 2$. Also $\sum_{n=2}^{\infty}\frac{1}{n}$ diverges as $p$-series. So we have that $\sum_{n=2}^{\infty}\frac{(x+2)^n}{2^n\ln n}$ diverges at $x = 0$.

For $x = -4$ we have

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}$$

so we apply the alternating series test. Since $\ln x$ is an increasing function $\frac{1}{\ln x}$ is a decreasing function and so $a_{n+1} < a_n$ for all $n \geq 2$. We also know $\lim_{n\to\infty}\ln n = \infty$ and so $\lim_{n\to\infty}\frac{1}{\ln n} = 0$. So this is a convergent alternating series.

Therefore, the interval of convergence is $-4 \leq x < 0$ or $[-4, 0)$. □

**Problem 10** (6 Points)**.** Find the Taylor series for $\frac{1}{x}$ about 1.

*Proof.* Successively taking derivatives we see that

$$f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}$$

So we get that

$$c_n = \frac{f^{(n)}(x)}{n!} = (-1)^n.$$

Therefore, the series is

$$\sum_{n=0}^{\infty}(-1)^n(x-1)^n.$$

□

**Problem 11** (6 Points)**.**

(a) Find a formula for the finite sum $\displaystyle\sum_{k=1}^{n}\left[\frac{k-1}{2k-1}-\frac{k}{2k+1}\right]$. (Hint: Write out a few terms.)

(b) Find $\displaystyle\sum_{k=1}^{\infty}\left[\frac{k-1}{2k-1}-\frac{k}{2k+1}\right]$.

*Proof.*

(a) We have the telescoping sum

$$\sum_{k=1}^{n}\left[\frac{k-1}{2k-1}-\frac{k}{2k+1}\right] = \left(0-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{2}{5}\right)+\left(\frac{2}{5}-\frac{3}{7}\right)+\cdots+\left(\frac{n-1}{2n-1}-\frac{n}{2n+1}\right)$$
$$= -\frac{n}{2n+1}.$$

(b) Since the sum of a series is the limit of its sequence of partial sums we have

$$\sum_{k=1}^{\infty}\left[\frac{k-1}{2k-1}-\frac{k}{2k+1}\right] = \lim_{n\to\infty} -\frac{n}{2n+1} = -\frac{1}{2}.$$

□

**Problem 12** (10 Points)**.** Use power series to estimate $\int_0^{1/2} \frac{\ln(1+x)}{x} dx$ to within 1/100.

*Proof.* Recall that the MacLaurin series for $\ln(1+x)$ is given by

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} = x - \frac{x^2}{2}+\frac{x^3}{3}-\cdots$$

so we have

$$\frac{\ln(1+x)}{x} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{n-1}}{n} = 1-\frac{x}{2}+\frac{x^2}{3}-\cdots.$$

Evaluating the integral yields

$$\int_0^{1/2}\frac{\ln(1+x)}{x} = \left(\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n^2}\right)\Bigg|_0^{1/2} \left(\sum_{n=1}^{\infty} x - \frac{x^2}{4}+\frac{x^3}{9}-\cdots\right)\Bigg|_0^{1/2}$$
$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2^n n^2}$$
$$= \frac{1}{2}-\frac{1}{16}+\frac{1}{72}-\frac{1}{400}+\cdots.$$

Since this is an alternating series, the remainder $|R_n| \leq a_{n+1}$ so we need the first 3 terms which is

$$\int_0^{1/2}\frac{\ln(1+x)}{x} \approx \frac{1}{2}-\frac{1}{16}+\frac{1}{72}.$$

□